Higher dimensional Reidemeister torsion invariants for cusped hyperbolic 3-manifolds

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Abstract

For an oriented finite volume hyperbolic 3-manifold M with a fixed spin structure η, we consider a sequence of invariants $\{\mathcal{T}_n(M;\eta)\}$. Roughly speaking, $\mathcal{T}_n(M;\eta)$ is the Reidemeister torsion of M respect to the representation given by the composition of the lift of the holonomy representation defined by η, and the n-th dimensional irreducible complex representation of $\mathrm{SL}(2,\mathbf{C})$. In the present work, we focus on two aspects of this invariant: its asymptotic behaviour and its relationship with the complex length spectrum of the manifold. Concerning the former, we prove that for suitable spin structures, $\log|\mathcal{T}_n(M;\eta)| \sim -n^2\frac{\mathrm{Vol}\,M}{4\pi}$, extending thus the result obtained by W. Müller for the compact case in [Mül]. Concerning the latter, we prove that the sequence $\{|\mathcal{T}_n(M;\eta)|\}$ determines the complex length spectrum of the manifold up to conjugation.

1 Introduction

Let M be an oriented complete hyperbolic 3-manifold. Consider its holonomy representation,

$$\mathrm{Hol}_M\colon \pi_1(M,p) \to \mathrm{Isom}^+\,\mathbf{H}^3 \cong \mathrm{PSL}(2,\mathbf{C}).$$

It is known that Hol_M can be lifted to $\mathrm{SL}(2,\mathbf{C})$; moreover, the set of such lifts is in canonical bijection with the set of spin structures on M, see [Cul86] and Section 2. In the sequel, this identification will be tacitly understood. We introduce the following definition.

Definition. A *spin hyperbolic* 3-manifold is a pair (M,η) where M is an oriented hyperbolic 3-manifold and a spin structure on M. The holonomy representation of (M,η) is defined as the lift to $\mathrm{SL}(2,\mathbf{C})$ of Hol_M given by η.

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Consider the representation ρ_n obtained by composing $\mathrm{Hol}_{(M,\eta)}$ with the n-th dimensional irreducible complex representation of $\mathrm{SL}(2,\mathbf{C})$; we have then

$$\rho_n \colon \pi_1(M,p) \to \mathrm{SL}(n,\mathbf{C}).$$

We would like to define the Reidemeister torsion $\tau(M;\rho_n)$ of M respect to ρ_n. (Since the image of ρ_n is contained in $\mathrm{SL}(n;\mathbf{C})$, the torsion has only a sign indeterminacy, that is, $\tau(M;\rho_n) \in \mathbf{C}^*/\{\pm 1\}$, see [Mil66],[Por97].) In order to do this, we need either ρ_n to be acyclic (i.e. $\mathrm{H}^*(M;\rho_n) = 0$), or to choose a basis for $\mathrm{H}^*(M;\rho_n)$.

Remark. We will follow the definition of the Reidemeister torsion used by Milnor in [Mil66]. The reason to do this is because of our main external results are from [Mül], which also uses this definition. The other common definition yields the inverse of ours.

An important case for which ρ_n is acyclic is when M is compact (this is a particular case of Raghunathan's cohomology vanishing Theorem, see Section 2). This case has been considered by W. Müller in [Mül].

For a cusped manifold M, the representation ρ_n is not acyclic in general; in fact, the group $\mathrm{H}^1(M;\rho_{2k+1})$ is never trivial for $k > 0$, as we will prove in Section 2. Therefore, we need to choose bases in (co)homology in order to define $\tau_n(M;\rho_n)$. Obviously, if we want an invariant of the manifold, these bases must be chosen in a somehow canonical way. Unfortunately, we do not know if this is possible. Nevertheless, we will prove in Proposition 2.2 that at least a canonical family of bases does exist; more concretely, given non-trivial cycles $\{\theta_i\}$ in $\mathrm{H}_1(\partial\overline{M};\mathbf{Z})$, one for each connected component of $\partial\overline{M}$, there is a canonical family of bases of $\mathrm{H}^*(M;\rho_n)$ such that any member of this family yields the same Reidemeister torsion, say $\tau(M;\rho_n;\{\theta_i\})$. Moreover, for $k > 1$ the following quotients are independent of the choices $\{\theta_i\}$,

$$\begin{aligned} \mathcal{T}_{2k+1}(M,\eta) &:= \frac{\tau(M;\rho_{2k+1};\{\theta_i\})}{\tau(M;\rho_3;\{\theta_i\})} \in \mathbf{C}^*/\{\pm 1\}, \\ \mathcal{T}_{2k}(M,\eta) &:= \frac{\tau(M;\rho_{2k};\{\theta_i\})}{\tau(M;\rho_2;\{\theta_i\})} \in \mathbf{C}^*/\{\pm 1\}. \end{aligned}$$

Therefore, for all $n \geq 4$, $\mathcal{T}_n(M,\eta)$ is an invariant of (M,η). Notice that for n odd $\mathcal{T}_n(M,\eta)$ is independent of the spin structure (this is an immediate consequence of the fact that an odd dimensional irreducible complex representation of $\mathrm{SL}(2,\mathbf{C})$ factors through $\mathrm{PSL}(2,\mathbf{C})$), and hence it will be denoted simply by $\mathcal{T}_{2k+1}(M)$.

The invariant $\mathcal{T}_n(M,\eta)$ will be called the *normalized n-th dimensional Reidemeister torsion.* The purpose of this work is its study.

Remark. It is possible to assign to $\mathcal{T}_n(M,\eta)$ a well defined sign: if n is pair, this can be done for $\tau_n(M;\rho_n)$ (see Turaev's book [Tur01]); and if n is odd, this can be done taking into account that, roughly speaking, the sign indeterminacy of $\tau_n(M;\rho_n)$ is the same as $\tau_3(M;\rho_n)$.

In spite of this, we will work always up to sign, as our main results concerns the modulus of $\mathcal{T}_n(M,\eta)$.

Let us summarize some of the results that we have obtained. To avoid technicalities in this introduction, we will restrict ourselves to the odd dimensional case. The results in the even dimensional case need the additional hypothesis on the spin structure of being acyclic, see Section 2.

Our main result concerns the asymptotic behaviour of the sequence $\{\mathcal{T}_{2k+1}(M)\}$.

Theorem 1.1. *Let M be an oriented complete finite volume hyperbolic 3-manifold. Then*

$$\lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k+1}(M)|}{(2k+1)^2} = -\frac{\mathrm{Vol}(M)}{4\pi}.$$

For M compact, this result has been established by W. Müller in [Mül]. To explain our approach to the above theorem, we must say some words about Müller's result.

Let us assume that M is compact. According to Müller's Theorem about the equivalence between Reidemeister torsion and Ray-Singer torsion for *unimodular* representations (see [Mül93]), we have

$$|\tau(M;\rho_n)| = \mathrm{Tor}(M;\rho_n),$$

where $\mathrm{Tor}(M;\rho_n)$ is the Ray-Singer torsion of M and ρ_n. For a manifold of negative curvature and a unitary representation ρ, D. Fried established in [Fri86] and [Fri95] a deep relationship between $\mathrm{Tor}(M,\rho_n)$ and the twisted Ruelle zeta function. Let us recall that the twisted Ruelle zeta function of M and ρ is formally defined by

$$R_\rho(s) = \prod_{\varphi\in\mathcal{PC}(M)} \det\left(\mathrm{Id} - \rho_n(\varphi)e^{-sl(\varphi)}\right), \tag{1}$$

where $\mathcal{PC}(M)$ denotes the set of oriented prime closed geodesics on M, and $l(\varphi)$ the length of φ (we are using the identification between $\mathcal{PC}(M)$ and the set of hyperbolic conjugacy classes of $\pi_1 M$, so the expression appearing inside the product makes sense). D. Fried proved that, for any representation ρ, $R_\rho(s)$ admits a meromorphic extension to the whole plane and, if ρ is assumed to be acyclic and unitary, then $|R_\rho(0)| = \mathrm{Tor}(M,\rho_n)^2$. The works of Bröcker [Brö98] and Wotzke [Wot08] showed that a similar result also holds for a compact hyperbolic manifold and representations of its fundamental group arising from representations of $\mathrm{Isom}^+\mathbf{H}^n$. In our particular case, the result is the following one, see also [Mül].

Theorem 1.2 (Wotzke, [Wot08])**.** *Let (M,η) be a compact spin hyperbolic 3-manifold. Then, for $n>1$, $R_{\rho_n}(s)$ admits a meromorphic extension to the whole complex plane and*

$$|R_{\rho_n}(0)| = \mathrm{Tor}(M;\rho_n)^2.$$

Bröcker proved in [Brö98] a functional equation for $R_{\rho_n}(s)$ involving the volume of the manifold. Using this equation and other related material, Müller has established in [Mül] the following formula for $|\tau(M;\rho_n)|$ which involves the volume of the closed manifold M and some related Ruelle zeta functions $R_k(s)$, which we will define in Section 5.

$$\log\left|\frac{\tau(M,\rho_{2k+1})}{\tau(M,\rho_5)}\right| = \sum_{j=3}^{k}\log|R_{2j}(j)| - \frac{1}{\pi}\operatorname{Vol}M\,(k(k+1)-6)\,. \tag{2}$$

One of the advantages of this formula is that the Ruelle zeta functions $R_k(s)$ are evaluated inside the corresponding region of convergence, and hence they have a nice expression similar to that of equation 1. The result about the asymptotics of the torsion is then derived by showing that the sum appearing in the right hand side of equation 2 is uniformly bounded on k.

It is worth saying that in adapting Müller's proof to the non-compact case, some difficulties arise, the main one being the fact that the Ray-Singer torsion is only defined for compact manifolds. Nevertheless, the terms appearing in equation 2 make sense also for cusped manifolds, and hence this equation is susceptible of being true also for such manifolds; in Section 5, we will prove that this indeed the case. Roughly speaking, our proof will consist in approximating the manifold M by the compact manifolds $\{M_{p/q}\}$ obtained by performing Dehn fillings on M. A formula relating $\mathcal{T}_{2k+1}(M)$ and $\mathcal{T}_{2k+1}(M_{p/q})$ will be given in Section 3 using a Mayer-Vietoris argument. As a by-product we will obtain the following result.

Proposition 1.3. *Let M be an oriented complete hyperbolic 3-manifold of finite volume with l cusps. Then the accumulation point set of $\{\mathcal{T}_{2k+1}(M_{p/q})\}$ is the segment joining the origin and the point $2^{2(k-1)l}\mathcal{T}_{2k+1}(M)$.*

The other thing we must take into account concerns the limit of the Ruelle zeta functions of the manifolds $M_{p/q}$ as (p,q) goes to infinity. Our main tool to deal with this will be the continuity of the complex length spectrum, which we briefly discuss now.

Definition. The *(prime) complex length spectrum* of M, denoted as $\mu_{\rm sp}\,M$, is the measure on $\mathbf{C}$ defined by

$$\mu_{\rm sp}\,M = \sum_{\varphi\in\mathcal{PC}(M)} \delta_{e^{\lambda(\varphi)}}\,,$$

where λ is the complex length function, and δ_x is the Dirac measure centered at x. In other words, $\mu_{\rm sp}\,M$ is the image measure of the counting measure in $\mathcal{PC}(M)$ under the exponential of the complex length function.

Remark. The complex length spectrum is usually regarded as a collection of complex numbers and multiplicities. This is of course equivalent to our definition; however, we think that to regard it as a measure puts some results in a natural context.

We can consider the prime complex length spectrum as a map from $\mathcal{M}$, the set of complete oriented hyperbolic 3-manifolds of finite volume, to $M(\mathbf{C} \setminus \overline{D})$, the set of measures on the exterior of the unit disc $\overline{D}$. Both spaces are endowed with natural topologies: the former with the geometric topology, and the latter with the topology of the weak convergence. Using standard techniques of hyperbolic geometry we will prove the continuity of this map in Section 4.

Theorem 1.4. *The map $\mu_{\mathrm{sp}} \colon \mathcal{M} \to M(\mathbf{C} \setminus \overline{D})$ which assigns to every finite volume complete oriented hyperbolic 3-manifold its complex length spectrum is continuous.*

With this formalism, equation 2 can be expressed in terms of the complex length spectrum measure. Using some complex analysis, we will deduce in Section6 that if we know all the values $\{|\mathcal{T}_{2k+1}(M)|\}_{k\geq N}$, for some $N \geq 1$, then we also know the values of the following integrals

$$M_k = \int_{|z|>1} (z^{-k} + \bar{z}^{-k}) \,\mathrm{d}\, \mu_{\mathrm{sp}} M(z), \quad k \geq N.$$

Using the Cauchy transform we will see that for this kind of measures this information is enough to recover the measure up to conjugation, that is, we do not know $\mu_{\mathrm{sp}} M$, but $\mu_{\mathrm{sp}} M + \overline{\mu_{\mathrm{sp}} M}$, where $\mu_{\mathrm{sp}} M$ denotes the image measure of $\mu_{\mathrm{sp}} M$ under complex conjugation. As a consequence, we will obtain the following result.

Theorem 1.5. *Let M be an oriented complete hyperbolic 3-manifold of finite volume. For all N, the sequence of values $\{|\mathcal{T}_{2k+1}(M)|\}_{k\geq N}$ determines the complex length spectrum of M up to conjugation.*

As a particular, if M admits an orientation reversing isometry (this is for instance the case of the figure eight knot exterior), then $\overline{\mu_{\mathrm{sp}} M} = \mu_{\mathrm{sp}} M$, and hence the sequence $\{|\mathcal{T}_{2k+1}(M)|\}_{k\geq N}$ determines the complex length spectrum completely.

Using Wotzke's Theorem we obtain the following corollary.

Corollary 1.6. *Let M be an oriented* compact *hyperbolic 3-manifold. The knowledge of $|\mathcal{T}_{2k+1}(M)|$ for all $k \geq N \geq 1$ is equivalent to the knowledge of the complex length spectrum of M up to conjugation.*

This article is organized as follows. In Section 2 we discuss some issues concerning the definition of the normalized torsion for cusped manifolds, specially the choice of bases for $\mathrm{H}^n(\partial\overline{M}; \rho_n)$. In Section 3, we study the behaviour of the normalized torsion under Dehn filling; this is done by using the Mayer-Vietoris formula for the Reidemeister torsion. In Section 4, we prove the continuity of the complex length spectrum measure; the techniques used there are standard in the context of 3-dimensional hyperbolic geometry (thick-thin decomposition, the geometric topology, Thurston's hyperbolic Dehn filling Theorem,...). In Section 5, we establish the asymptotic behaviour of the normalized torsion for cusped manifolds. Finally, in Section 6, we study the relationship between the complex length spectrum measure and the normalized Reidemeister torsion; we formulate this question in analytical terms, and then we solve it using techniques from complex analysis such as the Cauchy transform.

2 Higher dimensional Reidemeister torsion

Let (M, η) be a spin hyperbolic 3-manifold. We are interested in representations of $\pi_1(M,p)$ that are compositions of $\mathrm{Hol}_{(M,\eta)}$ and an *irreducible finite dimensional complex* representation of $\mathrm{SL}(2,\mathbf{C})$. The classification of these latter representations is well known: for every positive integer n there exists exactly one n-th dimensional complex irreducible representation V_n of $\mathrm{SL}(2,\mathbf{C})$, which is isomorphic to $\mathrm{Sym}^{n-1} V_2$, the $(n-1)$-th symmetric power of the standard representation $V_2 \cong \mathbf{C}^2$. We will use very often the fact that for all n there exists an invariant non-degenerated bilinear map on V_n,

$$\phi \colon V_n \times V_n \to V_n.$$

Moreover, ϕ is, up to homothety, the symmetric power of the determinant of V_2.

Let us denote by ρ_n the composition of $\mathrm{Hol}_{(M,\eta)}$ with the representation V_n just defined. Thus we have,

$$\rho_n \colon \pi_1(M,p) \to \mathrm{SL}(n,\mathbf{C}).$$

Notice that for n odd the representation V_n factors through $\mathrm{PSL}(2,\mathbf{C})$, and hence ρ_n is independent of the spin structure.

Next, we would like to define the Reidemeister torsion of M respect to the representation ρ_n. In order to do that, we need to show that M is ρ_n-acyclic, namely, that the groups $\mathrm{H}^*(M;\rho_n)$ are all trivial. If it does not happen, then we can still define the Reidemeister torsion of M respect to a fixed basis on (co)homology.

If M is compact, then, as a particular case of Raghunathan's vanishing Theorem [Rag65], the cohomology groups $\mathrm{H}^*(M;\rho_n)$ are all trivial, see also [BW80]. On the other hand, if M is non-compact, these groups need not to be trivial; nevertheless, the same techniques used by Raghunathan in [Rag65] –more concretely, those developed by Matsushima-Murakami in [MM63]– lead to the following result, see [MFP].

Theorem 2.1 ([MFP]). *Let (M,η) be a complete spin hyperbolic 3-manifold. Assume that M is the interior of compact manifold $\overline{M}$ (i.e. M is topologically finite). Then for $n \geq 2$ the inclusion $\partial\overline{M} \subset \overline{M}$ induces an injection*

$$\mathrm{H}^1(M;\rho_n) \hookrightarrow \mathrm{H}^1(\partial\overline{M};\rho_n),$$

with $\dim_{\mathbf{C}} \mathrm{H}^1(M;\rho_n) = \frac{1}{2}\dim_{\mathbf{C}} \mathrm{H}^1(\partial\overline{M};\rho_n)$*, and an isomorphism*

$$\mathrm{H}^2(M;\rho_n) \to \mathrm{H}^2(\partial\overline{M};\rho_n).$$

Roughly speaking, this theorem says that the cohomology information of M in the local system given by ρ_n comes from the boundary. For $n = 3$, V_n is the adjoint representation of $\mathrm{PSL}(2,\mathbf{C})$, and $\mathrm{H}^1(M;\rho_3)$ has a well known interpretation in terms of deformation theory.

In what follows we will restrict ourselves to finite-volume manifolds. Thus M is the interior of a compact manifold $\overline{M}$ such that

$$\partial\overline{M} = T_1 \cup \cdots \cup T_l,$$

where each connected component T_i is homeomorphic to a torus $S^1 \times S^1$. Notice that in this case Theorem 2.1, for $n = 3$, agrees with the fact that a deformation of the complete hyperbolic structure of M is determined by the deformations of the cusps.

The dimensions of the cohomology groups $\mathrm{H}^*(\partial\overline{M};\rho_n)$ are easily determined once we know the dimension of $\mathrm{H}^0(\partial\overline{M};\rho_n)$ (this follows from Poincaré duality and an Euler characteristic argument). Interpreting this latter group in terms of fixed vectors, it is easy to deduce that $\mathrm{H}^0(T_i;\rho_{2k+1})$ has (complex) dimension 1 for $k > 0$, see Subsection 2.2. Therefore, Theorem 2.1 implies that $\mathrm{H}^0(M;\rho_{2k+1})$ has dimension l. For a positive even n, the situation is a little bit more subtle, and the cohomology groups $\mathrm{H}^0(T_i;\rho_n)$ may be trivial or not depending on the chosen spin structure η.

Definition. Let M be an oriented complete hyperbolic 3-manifold of finite volume. A spin structure η on M will be called *acyclic* if the cohomology groups $\mathrm{H}^*(M;\mathrm{Hol}_{(M,\eta)})$ are all trivial, or, equivalently, if $\mathrm{H}^0(\partial\overline{M};\mathrm{Hol}_{(M,\eta)})$ is trivial.

Remark. If η is acyclic then the groups $\mathrm{H}^*(M;\rho_{2k})$ are also trivial for $k > 1$. In that case, the Reidemeister torsion $\tau(M;\rho_{2k}) \in \mathbf{C}^*/\{\pm 1\}$ is defined. It must be pointed out that there always exist acyclic spin structures, see Proposition 2.7 and its corollaries.

If it happens that ρ_n is not acyclic, then we need to specify bases in (co)homology in order to define the Reidemeister torsion; it turns out that these bases are better characterized in homological terms rather than in cohomological ones. The proof of the following proposition will be given in Subsection 2.2.

Proposition 2.2. *Let $n > 0$. For each connected boundary component T_i of M such that $\mathrm{H}^0(T_i;\rho_n)$ is not trivial, fix a non-trivial cycle $\theta_i \in \mathrm{H}_1(T_i;\mathbf{Z})$. Then there exists a canonical family of bases for the homology groups $\mathrm{H}_*(M;\rho_n)$ such that any basis of this family determines the same Reidemeister torsion, say $\tau(M;\rho_n;\{\theta_i\})$. Moreover, for all $k > 0$ the following quantities are independent of $\{\theta_i\}$,*

$$\begin{aligned} \mathcal{T}_{2k+1}(M,\eta) &:= \frac{\tau(M;\rho_{2k+1};\{\theta_i\})}{\tau(M;\rho_3;\{\theta_i\})} \in \mathbf{C}^*/\{\pm 1\}, \\ \mathcal{T}_{2k}(M,\eta) &:= \frac{\tau(M;\rho_{2k};\{\theta_i\})}{\tau(M;\rho_2;\{\theta_i\})} \in \mathbf{C}^*/\{\pm 1\}. \end{aligned}$$

Definition. Let (M,η) be a complete spin hyperbolic 3-manifold of finite volume. For $n \geq 4$, the invariant $\mathcal{T}_n(M,\eta)$ defined in the above proposition is called the *n-th normalized Reidemeister torsion* of the spin manifold (M,η). If $n = 2k+1$ is odd, $\mathcal{T}_{2k+1}(M;\eta)$ is independent of η, and will be denoted by $\mathcal{T}_{2k+1}(M)$.

The rest of this section is organized as follows. The first subsection is a review of the relation between spin structures and lifts of the holonomy representation. This is done in a more or less self-contained way. We prove in that subsection the existence of acyclic spin structures. The second subsection is devoted to the analysis of the groups $\mathrm{H}_*(M;\rho_n)$ in the case of course that they are not trivial. The aim of that subsection is to prove Proposition 2.2.

2.1 Lifts of the holonomy representation

Let M be a connected oriented hyperbolic 3-manifold which is not necessarily complete. The aim of this subsection is to review the relation between spin structures on M and lifts of its holonomy representation to $\mathrm{SL}(2,\mathbf{C})$. We will use the following definition of a spin structure, see [Kir89]. The SO(3)-principal bundle of orthonormal positively oriented frames of M is denoted by $P_{\mathrm{SO}(3)}M$.

Definition. A spin structure on M is a (double) cover of $P_{\mathrm{SO}(3)}M$ by a Spin(3)-principal bundle over M.

The above definition is equivalent to say that a spin structure on M is a double cover of $P_{\mathrm{SO}(3)}M$ such that the preimage of any fiber of $P_{\mathrm{SO}(3)}M$ is connected. It can be checked then that there is a natural identification between the set of spin structures on M and the set

$$\left\{\alpha \in \mathrm{H}^1(P_{\mathrm{SO}(3)}M;\mathbf{Z}/2\mathbf{Z}) \mid i^*(\alpha) = 1 \in \mathrm{H}^1(\mathrm{SO}(3);\mathbf{Z}/2\mathbf{Z})\right\}.$$

On the other hand, the hyperbolic structure of M defines a canonical flat $\mathrm{Isom}^+\mathbf{H}^3$-principal bundle over M, see [Thu97]. Let us recall how it is defined. Let $\mathbf{H}^3$ be hyperbolic space of dimension 3 with a fixed orientation. Consider an $(\mathrm{Isom}^+\mathbf{H}^3,\mathbf{H}^3)$-atlas on M defining the hyperbolic structure. Thus we have local charts $\phi_i\colon U_i \to \mathbf{H}^3$ covering M such that the changes of coordinates are restrictions of orientation preserving isometries of $\mathbf{H}^3$. We can assume that the local charts preserve the fixed orientations on both M and $\mathbf{H}^3$. Let ψ_{ij} be the change of coordinates from (ϕ_j, U_j) to (ϕ_i, U_i), that is,

$$\psi_{ij}\colon U_i \cap U_j \to \mathrm{Isom}^+\mathbf{H}^3, \quad \psi_{ij}\circ\phi_j = \phi_i.$$

The analyticity of the elements of $\mathrm{Isom}^+\mathbf{H}^3$ implies that ψ_{ij} is a locally constant map. Since these maps also satisfy the cocycle condition $\psi_{ij}\circ\psi_{jk} = \psi_{ik}$, they define a flat $\mathrm{Isom}^+\mathbf{H}^3$-principal bundle over M,

$$\mathrm{Isom}^+\mathbf{H}^3 \to P_{\mathrm{Isom}^+\mathbf{H}^3}M \xrightarrow{\pi} M.$$

Let us fix a base point $p \in M$. Given $u \in P_{\mathrm{Isom}^+\mathbf{H}^3}M$ with $\pi(u) = p$, it makes sense to consider the holonomy representation of this principal bundle,

$$\mathrm{Hol}_u\colon \pi_1(M,p) \to \mathrm{Isom}^+\mathbf{H}^3.$$

Recall that, by definition, if $\sigma\colon [0,1] \to M$ is a loop based at p, $\mathrm{Hol}_u(\sigma)$ is the unique element of $\mathrm{Isom}^+ \mathbf{H}^3$ such that

$$\tilde{\sigma}(1) \cdot \mathrm{Hol}_u(\sigma) = \tilde{\sigma}(0),$$

where $\tilde{\sigma}(t)$ is the horizontal lift of $\sigma(t)$ starting at u. It can be checked that this holonomy is, up to a conjugation, the same as the one given in terms of the developing map. In other words, for some suitable initial choices, we have $\mathrm{Hol}_u = \mathrm{Hol}_M$.

Proposition 2.3. *There is a canonical one-to-one correspondence between the following sets:*

1. *The set of covers of* $P_{\mathrm{PSL}(2,\mathbf{C})}M$ *by* $\mathrm{SL}(2,\mathbf{C})$*-principal bundles over* M*.*

2. *The set of lifts of* Hol_M *to* $\mathrm{SL}(2,\mathbf{C})$*.*

Proof. Let us assume that we have chosen base points $p \in M$, $u \in P_{\mathrm{PSL}(2,\mathbf{C})}M$ with $\pi(u) = p$, such that $\mathrm{Hol}_u = \mathrm{Hol}_M$. Let $P_{\mathrm{SL}(2,\mathbf{C})}M$ be an $\mathrm{SL}(2,\mathbf{C})$-principal bundle over M covering $P_{\mathrm{PSL}(2,\mathbf{C})}M$. Take one of the two points $\tilde{u} \in P_{\mathrm{SL}(2,\mathbf{C})}M$ that projects to u, and consider the corresponding holonomy representation $\mathrm{Hol}_{\tilde{u}}$. It is clear that $\mathrm{Hol}_{\tilde{u}}$ is a lift of Hol_u; moreover, it is independent of the choice of the base point $\tilde{u}$, since the other choice is obtained by conjugating it by $-\mathrm{Id} \in \mathrm{SL}(2,\mathbf{C})$. This gives a well defined correspondence between the set of covers of $P_{\mathrm{PSL}(2,\mathbf{C})}M$ by $\mathrm{SL}(2,\mathbf{C})$-principal bundles over M and the set of lifts of Hol_M to $\mathrm{SL}(2,\mathbf{C})$. Since we can recover the flat bundle from its holonomy representation, this correspondence is one-to-one. □

Next, we want to embed the frame bundle $P_{\mathrm{SO}(3)}M$ into $P_{\mathrm{PSL}(2,\mathbf{C})}M$. To that end, identify $\mathrm{PSL}(2,\mathbf{C})$ with $P_{\mathrm{SO}(3)}\mathbf{H}^3$ by fixing a positively oriented frame $R_O \in P_{\mathrm{SO}(3)}\mathbf{H}^3$. Notice that it gives a concrete embedding of $\mathrm{SO}(3)$ into $\mathrm{PSL}(2,\mathbf{C})$. Now let $u \in P_{\mathrm{PSL}(2,\mathbf{C})}M$ and $p = \pi(u)$. Take a local chart (ϕ_j, U_j) of the hyperbolic structure with $p \in U_j$. This gives a local trivialization $U_j \times G$ of $P_{\mathrm{PSL}(2,\mathbf{C})}M$. Respect to this trivialization, the point u is written as a pair $(p, g) \in U_j \times \mathrm{PSL}(2,\mathbf{C})$. We will say that *$u$ is based at $p \in M$* , if $g \in \mathrm{PSL}(2,\mathbf{C}) \cong P_{\mathrm{SO}(3)}\mathbf{H}^3$ is a frame based at $\phi_j(p)$. It can be checked that this definition does not depend on the choice of the local chart (ϕ_j, U_j). The following set is canonically identified with the bundle of positively oriented frames in M,

$$\{u \in P_{\mathrm{Isom}^+\mathbf{H}^3}M \mid u \text{ is a frame based at } \pi(u)\} \cong P_{\mathrm{SO}(3)}M.$$

This gives a concrete embedding $P_{\mathrm{SO}(3)}M \hookrightarrow P_{\mathrm{PSL}(2,\mathbf{C})}M$ (in other words, we have obtained an explicit reduction of the structural group respect to the fixed embedding $\mathrm{SO}(3) \subset \mathrm{PSL}(2,\mathbf{C})$). Although the embedding just defined depends on the choices done, it must be pointed out that its homotopy class does not.

Proposition 2.4. *There is a canonical one-to-one correspondence between the following sets:*

1. *The set of covers of* $P_{\mathrm{PSL}(2,\mathbf{C})}M$ *by* $\mathrm{SL}(2,\mathbf{C})$*-principal bundles over* M*.*

2. *The set of spin structures on M.*

Proof. As we have observed, the set of spin structures on M is canonically identified with

$$\left\{\alpha \in \mathrm{H}^1(P_{\mathrm{SO}(3)}M; \mathbf{Z}/2\mathbf{Z}) \mid i^*(\alpha) = 1 \in \mathrm{H}^1(\mathrm{SO}(n); \mathbf{Z}/2\mathbf{Z})\right\}.$$

The same argument shows that the set of covers of $P_{\mathrm{PSL}(2,\mathbf{C})}M$ by $\mathrm{SL}(2,\mathbf{C})$-principal bundles over M is identified with

$$\left\{\alpha \in \mathrm{H}^1(P_{\mathrm{PSL}(2,\mathbf{C})}M; \mathbf{Z}/2\mathbf{Z}) \mid i^*(\alpha) = 1 \in \mathrm{H}^1(\mathrm{SL}(2,\mathbf{C}); \mathbf{Z}/2\mathbf{Z})\right\}.$$

The result then follows from the fact that the map $P_{\mathrm{SO}(3)}M \hookrightarrow P_{\mathrm{Isom}^+\mathbf{H}^3}M$ defined above, whose homotopy class is canonical, is a homotopy equivalence as $\mathrm{SO}(3) \simeq \mathrm{PSL}(2,\mathbf{C})$. $\square$

Corollary 2.5. *The holonomy representation of a hyperbolic* 3*-manifold can be lifted to* $\mathrm{SL}(2,\mathbf{C})$*. The number of such lifts is* $|\,\mathrm{H}^1(M; \mathbf{Z}/2\mathbf{Z})|$*.*

Proof. An oriented 3-manifold admits $|\,\mathrm{H}^1(M; \mathbf{Z}/2\mathbf{Z})|$ different spin structures. $\square$

Next we want to prove the existence of acyclic spin structures on an oriented complete hyperbolic 3-manifold M of finite volume. Recall that a spin structure η is acyclic if and only if $\mathrm{H}^0(T_i; \mathrm{Hol}_{(M,\eta)})$ is trivial for each connected component T_i of the boundary. Let us fix T_i, and denote it as T^2. We can assume that T^2 is a horospheric cross-section, and that

$$\mathrm{Hol}_M(\pi_1 T^2) = \left\langle \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}\right] \right\rangle < \mathrm{PSL}(2,\mathbf{C}), \qquad \text{with } \operatorname{Im}\tau > 0.$$

Let $P_{\mathrm{SO}(3)}T^2 \subset P_{\mathrm{PSL}(2,\mathbf{C})}T^2$ be the restriction of $P_{\mathrm{SO}(3)}M \subset P_{\mathrm{PSL}(2,\mathbf{C})}M$ to T^2, and let Hol_{T^2} be the restriction of Hol_M to $\pi_1 T^2$.

Using the Euclidean structure of T^2 and the outward normal vector of T^2 we can construct a section s of the bundle $P_{\mathrm{SO}(3)}T^2$ that is canonical up to conjugation. Indeed, fix $p \in T^2$ and define $s(p) \in P_{\mathrm{SO}(3)}T^2$ as any frame based at p whose third component is equal to the outward normal vector at p; for $q \in T^2$, define $s(q)$ as the parallel transport (respect to the euclidean structure) of $s(p)$ along a curve joining p and q on T^2. This yields a well defined section which is canonical up to homotopy. Thus we have a canonical trivialization $P_{\mathrm{SO}(3)}T^2 = T^2 \times \mathrm{SO}(3)$, and hence a distinguished spin structure $T^2 \times \mathrm{Spin}(3)$. All the others spin structures arise as quotients of the form

$$\eta_\alpha = \left(\widetilde{T^2} \times \mathrm{Spin}(3)\right) / \pi_1 T^2,$$

where $\alpha \in \mathrm{H}^1(T^2; \mathbf{Z}/2\mathbf{Z}) = \mathrm{Hom}(\pi_1 T^2; \{\pm 1\})$, and $\sigma \in \pi_1 T^2$ acts on $\mathrm{Spin}(3)$ by multiplication by $\alpha(\sigma)\,\mathrm{Id}$. Therefore, spin structures of $P_{\mathrm{SO}(3)}T^2$ are in canonical one-to-one correspondence with $\mathrm{H}^1(T^2; \mathbf{Z}/2\mathbf{Z})$.

Therefore, given $\alpha \in \mathrm{H}^1(T^2; \mathbf{Z}/2\mathbf{Z})$ we have also a cover of $P_{\mathrm{PSL}(2,\mathbf{C})}T^2$ by an $\mathrm{SL}(2,\mathbf{C})$-principal bundle. The above discussion shows that this is given by

$$\left(\widetilde{T^2} \times \mathrm{SL}(2,\mathbf{C})\right)/\pi_1 T^2,$$

where $\sigma \in \pi_1 T^2$ acts on $\mathrm{SL}(2,\mathbf{C})$ by multiplication by $\alpha(\sigma)\,\mathrm{Id}$. Taking this into account, the following result is easily proved.

Lemma 2.6. *Let $\alpha \in \mathrm{H}^1(T^2; \mathbf{Z}/2\mathbf{Z}) = \mathrm{Hom}(\pi_1 T^2; \{\pm 1\})$, η_α the associated spin structure on T^2, and $\mathrm{Hol}_{(T^2,\eta_\alpha)}$ the corresponding lift of the holonomy representation. Then we have*

$$\alpha(\sigma) = \operatorname{sgn} \operatorname{trace} \mathrm{Hol}_{(T^2,\eta_\alpha)}(\sigma), \quad \textit{for all } \sigma \in \pi_1 T^2.$$

Now we can prove the existence of spin acyclic structures.

Proposition 2.7. *Let M be an oriented complete hyperbolic 3-manifold of finite volume. For each connected boundary component T_i take a closed simple curve γ_i. Then there exists a spin structure η on M such that*

$$\operatorname{trace} \mathrm{Hol}_{(M,\eta)}([\gamma_i]) = -2,$$

where $[\gamma_i]$ is the conjugacy class of $\pi_1(M,p)$ defined by γ_i.

Proof. Let N be the manifold obtained by performing a Dehn filling along each of the curves $\{\gamma_i\}$. Fix a spin structure η on N. We claim that the restriction of η to M gives the required spin structure.

Assume that γ is one of the curves γ_i, and that it is contained in a horospheric cross-section T^2. For the sake of concreteness, assume also that γ is a closed geodesic respect to the euclidean structure of T^2. Let $P_{\mathrm{Spin}(3)}T^2 \to P_{\mathrm{SO}(3)}T^2$ the corresponding Spin(3)-bundle over T^2 defined by η. Consider the canonical section $s\colon T^2 \to P_{\mathrm{SO}(3)}T^2$ constructed above using as starting frame one whose first vector is tangent to γ. Notice that the closed curve $s \circ \gamma$ can be lifted to $P_{\mathrm{Spin}(3)}T^2$ if and only if $\alpha(\gamma) = 1$. On the other hand, if $s \circ \gamma$ could be lifted to $P_{\mathrm{Spin}(3)}T^2$, then such lift could be extended to added disk that bounds γ (as there is no obstruction in doing it, $\pi_1 \mathrm{Spin}(3) = \{1\}$), and hence $s \circ \gamma$ could be extended to that disk, what is not possible by construction. Thus $\alpha(\gamma) = -1$, and the preceding lemma implies the result. $\square$

As a corollary of the proof of the above proposition, we obtain the following result.

Corollary 2.8. *Let η be a spin structure on M. Let $\gamma \subset \partial M$ be a simple closed curve non-homotopically trivial in ∂M, and M_γ be the manifold obtained by performing a Dehn filling along γ. Then η extends to a spin structure on M_γ if and only if $\mathrm{Hol}_{(M,\eta)}(\gamma)$ has trace -2.*

The following corollary of Proposition 2.7 gives a sufficient condition that guarantees the acyclicity of all spin structures.

Corollary 2.9. *Assume that for each connected boundary component T_i of M, the map $\mathrm{H}_1(T_i;\mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}_1(M;\mathbf{Z}/2\mathbf{Z})$ induced by the inclusion has non-trivial kernel. Then all spin structures on M are acyclic. In particular, if M has one cusp, all spin structures are acyclic.*

Proof. If the hypothesis holds, then for each T_i there exists a closed simple curve $\gamma_i \in T_i$ that is zero in $\mathrm{H}_1(M;\mathbf{Z}/2\mathbf{Z})$. Take a spin structure on M such that trace $\mathrm{Hol}_{(M,\eta)}([\gamma_i]) = -2$, for all γ_i. Now let η' be another spin structure on M, and $\alpha \in \mathrm{H}^1(M;\mathbf{Z}/2\mathbf{Z})$ the cohomology class relating η and η'. Then, using multiplicative notation, we have

$$\mathrm{Hol}_{(M,\eta')}(\gamma_i) = \alpha(\gamma_i)\,\mathrm{Hol}_{(M,\eta)}(\gamma_i).$$

Since $\alpha(\gamma_i) = 1$, $\mathrm{Hol}_{(M,\eta')}(\gamma_i)$ has trace -2, as we wanted to prove. The rest of the result follows from the fact that in any compact 3-manifold M the map $\mathrm{H}_1(\partial M;\mathbf{Z}/2\mathbf{Z}) \to \mathrm{H}_1(M;\mathbf{Z}/2\mathbf{Z})$ induced by the inclusion has a non-trivial kernel. $\square$

2.2 The homology groups $\mathrm{H}_*(M;\rho_n)$

The aim of this subsection is to prove Proposition 2.2 concerning the existence of a distinguished family of bases for the groups $\mathrm{H}_*(M;\rho_n)$.

We will use the following construction for the homology of a CW-complex X in the local system defined by a representation $\rho\colon \pi_1(X,p) \to \mathrm{GL}(V)$. Let $C_*(\widetilde{X};\mathbf{Z})$ denote the complex of singular chains on the universal covering, in which $\pi_1(X,p)$ acts on the left by deck transformations. Consider the right action of $\pi_1(X,p)$ on V, so that $\gamma \in \pi_1(X,p)$ maps $v \in V$ to $\rho(\gamma)^{-1}v$. Then $\mathrm{H}_*(X;\rho)$ is the homology of the following complex of $\mathbf{C}$-vector spaces,

$$\left(V_n \otimes_{\mathbf{Z}[\pi_1(X,p)]} C_*(\widetilde{X};\mathbf{Z}), \mathrm{Id}\otimes\partial_*\right).$$

We shall use the Kronecker pairing between homology and cohomology with twisted coefficients. In order to define it we need an invariant and non-degenerated bilinear map

$$\phi\colon V \times V \to \mathbf{C}.$$

We recall that in the case that X is a differentiable manifold, the Kronecker pairing can be defined at the level of chains and cochains as follows: the pairing between a smooth i-cycle $\sum_\theta v_\theta \otimes \theta$, with $\theta \in C_i(\widetilde{X};\mathbf{Z})$, $v_\theta \in V$ and a closed i-form $\sum_\omega \omega \otimes v_\omega$ is

$$\sum_{\theta,\omega} \int_\theta \phi(v_\theta, v_\omega)\,\omega \in \mathbf{C}.$$

It does not depend on the different choices, but on the respective classes in cohomology and homology.

Let (M,η) be a connected complete spin hyperbolic 3-manifold of finite volume. Let $n > 0$ be a fixed integer number, and consider the representation defined in the first subsection

$$\rho_n\colon \pi_1(M,p) \to \mathrm{SL}(n,\mathbf{C}).$$

The rest of this subsection is devoted to prove the following result from which Proposition 2.2 is immediately deduced. Before stating it, let us recall the following definition.

Definition. Let θ_1, θ_2 be two non-trivial cycles in a boundary component T_j of $\overline{M}$. Using the canonical identification between $\mathrm{H}_1(T_j;\mathbf{Z}) \cong \pi_1 T_j$, let us assume that

$$\mathrm{Hol}_M(\theta_i) = \left[\begin{pmatrix} 1 & a(\theta_i) \\ 0 & 1 \end{pmatrix}\right] \in \mathrm{PSL}(2,\mathbf{C})$$

for some $a(\theta_i) \in \mathbf{C}^*$, $i = 1,2$. Then define the *cusp shape* of the pair (θ_1, θ_2) as

$$\mathrm{cshape}(\theta_1, \theta_2) = \frac{a(\theta_1)}{a(\theta_2)}.$$

Notice that $\mathrm{cshape}(\theta_1, \theta_2)$ is well defined, because $a\colon \pi_1 T_j \to \mathbf{C}$ is unique up to homothety.

Proposition 2.10. *Let $T_1, \dots, T_r$ be the boundary components of $\overline{M}$ that are* not *ρ_n-acyclic. Let $G_j < \pi_1(M,p)$ be some fixed realization of the fundamental group of T_j as a subgroup of $\pi_1(M,p)$. For each T_j choose a non-trivial cycle $\theta_j \in \mathrm{H}_1(M;\mathbf{Z})$, and a non-trivial vector $w_j \in V_n$ fixed by $\rho_n(G_j)$. If $i_j\colon T_j \to M$ denotes the inclusion, then we have:*

1. *A basis for $\mathrm{H}_1(M;\rho_n)$ is given by*

$$(i_{1*}([w_1 \otimes \theta_1]), \dots, i_{r*}([w_r \otimes \theta_r]))\,.$$

2. *Let $[T_j] \in \mathrm{H}_2(T_j;\mathbf{Z})$ be a fundamental class of T_j. A basis for $\mathrm{H}_2(M;\rho_n)$ is given by*

$$(i_{1*}([w_1 \otimes T_1]), \dots, i_{r*}([w_r \otimes T_r]))\,.$$

Furthermore, if we choose different non-trivial cycles $\theta'_1, \dots, \theta'_r$ then

$$i_{j*}([w_j \otimes \theta_j]) = \mathrm{cshape}(\theta_j, \theta'_j)\, i_{j*}([w_j \otimes \theta'_j]).$$

The proof of the above proposition will be given at the end of this section.

2.2.1 The cohomology groups $\mathrm{H}^*(\partial\overline{M};\rho_n)$

Let T_j be a connected component of $\partial\overline{M}$, and U_j the corresponding cusp. It is well known that T_j can be identified with the set of rays contained in U_j, and that it endows T_j with a canonical similarity structure; in particular, T_j has a canonical holomorphic structure. Let us denote by $\pi_j\colon U_j \to T_j$ the canonical projection, which sends a point in U_j to ray it belongs to.

Let E_n be the flat vector bundle over $\overline{M}$ defined by the representation ρ_n. In order to compute $\mathrm{H}^*(T_i;\rho_n)$ we will interpret it as the cohomology of the de Rham complex

$(\Omega^*(T_i^2; E_n), d_\nabla)$, where d_∇ is the covariant differential defined by the flat connection. This complex is isomorphic to the complex $(\Omega^*(\widetilde{T}_i; V_n)^{\pi_1 T_i}, d)$ of equivariant V_n-valued differential forms with the usual exterior differential.

On the other hand, E_n is a holomorphic vector bundle respect to the holomorphic structure of T_i. Thus we have the following canonical decomposition,

$$\Omega^1(T_i; E_n) = \Omega^{1,0}(T_i; E_n) \oplus \Omega^{0,1}(T_i; E_n),$$

where $\Omega^{1,0}(T_i; E_n)$ and $\Omega^{0,1}(T_i; E_n)$ are the spaces of E_n-valued 1-forms of type $(1,0)$ and $(0,1)$ respectively. Let us denote as $\mathrm{H}^{r,s}(T_i; E_n)$ the projection of $\Omega^{r,s}(T_i; E_n) \cap \operatorname{Ker} d$ onto $\mathrm{H}^1(T_i; E_n)$, with $(r,s) = (0,1), (1,0)$.

Proposition 2.11. *Assume that* $\mathrm{H}^0(T_i; E_n) \neq 0$. *Then,*

$$\mathrm{H}^1(T_i; E_n) = \mathrm{H}^{0,1}(T_i; E_n) \oplus \mathrm{H}^{1,0}(T_i; E_n).$$

Proof. We can assume that for all $\gamma \in \pi_1 T_i$ we have,

$$\mathrm{Hol}_M(\gamma) = \left[\begin{pmatrix} 1 & a(\gamma) \\ 0 & 1 \end{pmatrix} \right] \in \mathrm{PSL}(2, \mathbf{C}).$$

This choice of the holonomy representation gives a complex coordinated z on $\widetilde{T}_i$. Identifying V_n with the space of $(n-1)$-th degree homogeneous polynomials in the variable X, Y, we define the following two forms on $\Omega^*(\widetilde{T}_i; V_n)$,

$$\alpha = d\bar{z} \otimes X^{n-1}, \quad \beta = dz \otimes (zX + Y)^{n-1}.$$

The action of $\gamma \in \pi_1 T_i$ on $P(X, Y)$ is given by

$$\gamma \cdot P(X, Y) = P(\epsilon X, \epsilon(a(\gamma) X + Y)),$$

where $\epsilon = \pm 1$ is the sign of the trace of γ determined by the lift of the holonomy representation. If n is odd, it is immediate to check that α and β are equivariant forms. On the other hand, if n is even, the condition that $\mathrm{H}^0(T_i; E_n)$ be not trivial is equivalent to say that $\mathrm{Hol}_{(M,\eta)}(\sigma)$ has trace 2 for all $\sigma \in \pi_1 T_i$; hence, $\epsilon = 1$, and the two forms are equivariant as well. Since they are closed forms, they define cohomology classes in $\mathrm{H}^1(T_i; E_n)$, and $[\alpha] \in \mathrm{H}^{0,1}(T_i; E_n)$ and $[\beta] \in \mathrm{H}^{1,0}(T_i; E_n)$. Therefore, to prove the result it remains to prove that $[\alpha]$ and $[\beta]$ are linearly independent, as $\dim_{\mathbf{C}} \mathrm{H}^1(T_i; \rho_m) = 2$. This last assertion is equivalent to say that $[\alpha] \wedge [\beta] \in \mathrm{H}^2(T^2; \mathbf{C})$ is not zero. A simple computation shows that

$$\alpha \wedge \beta = \phi(X^{n-1}, (zX + Y)^{n-1}) d\bar{z} \wedge dz = d\bar{z} \wedge dz,$$

where ϕ is the non-degenerate $\mathrm{SL}(2, \mathbf{C})$-invariant pairing of V_n. This shows that $[\alpha] \wedge [\beta]$ is not zero, and hence the two classes must be linearly independent. □

Next we want to characterize the image of the map induced by the inclusion

$$i^*\colon \mathrm{H}^1(\overline{M};E_n) \to \mathrm{H}^1(\partial\overline{M};E_n).$$

Although this description will not be complete, it will be enough to give bases for the homology groups $\mathrm{H}_*(M;\rho_n)$. Before analysing the general case, let us discuss briefly the case $n=3$.

The representation V_3 is the adjoint representation of $\mathrm{SL}(2;\mathbf{C})$, and the cohomology group $\mathrm{H}^*(M;E_3)$ has a geometrical interpretation in terms of infinitesimal deformations of the complete hyperbolic structure. The vector bundle E_3 is identified with the bundle of germs of Killing vector fields on M, and, with the same notation as in the proof of the above proposition, it can be checked that the global section X^2 corresponds to the vector field $\frac{\partial}{\partial z_j}$. With this description, the 1-form $d\bar{z}_j \otimes \frac{\partial}{\partial z_j}$ is $(0,1)$-form that takes values in the vector bundle of holomorphic fields. According to the theory of deformations of complex manifolds, this cohomology class describes the deformations of the holomorphic structure of T_j by deformations of the defining lattice; in particular, it gives a deformation of the euclidean structure through euclidean structures. On the other hand, a non-trivial deformation of the complete hyperbolic structure is encoded by a cohomology class $\omega \in \mathrm{H}^1(M;E_3)$, and $i^*(\omega)$ encodes the corresponding deformation of the similarity structure in each torus. Since this deformation cannot be through euclidean structures on all tori, then, for some T_j, the restriction of $i^*(\omega)$ to T_j can not be contained in $\mathrm{H}^{0,1}(T_j;E_3)$. This shows that there is the following decomposition,

$$\mathrm{H}^1(\partial\overline{M};E_3) = \operatorname{Im} i^* \bigoplus_{j=1}^{k} \mathrm{H}^{0,1}(T_j^2;E_3).$$

We would like to prove that the above decomposition holds also for $n\geq 2$. Unfortunately, we do not have an interpretation of the cohomology group $\mathrm{H}^1(M;E_n)$ in geometrical terms such as deformations. Thus we shall proceed in a different way. Our key tool will be Theorem 2.1 of [MFP], which states that, respect to a suitable hermitian metric on E_n, a class $\omega \in \mathrm{H}^1(M;E_n)$ cannot be represented by a square-integrable form. Let us recall the definition of such metric on E_n. Choose any $\mathrm{SU}(2)$-invariant hermitian metric $\langle\cdot,\cdot\rangle$ in V_n (we are considering $\mathrm{SU}(2)$ as a subgroup of $\mathrm{SL}(2,\mathbf{C})$). Identify $\mathbf{H}^3$ with $\mathrm{SL}(2,\mathbf{C})/\mathrm{SU}(2)$, and let $p\in\mathbf{H}^3$ be the class of the identity. Define a hermitian structure on the trivial vector bundle $\mathbf{H}^3\times V_n$ by

$$\langle (q,w_1),(q,w_2)\rangle_q = \langle g w_1, g w_2\rangle, \quad \text{where } g\cdot q = p.$$

It is immediately to check that it is well defined and induces a hermitian metric on the vector bundle $E_n = \mathbf{H}^3 \times_{\pi_1(M,p)} V_n$.

Lemma 2.12. *Assume that $\mathrm{H}^0(T_j;E_n)\neq 0$, and let $\pi_j\colon U_j\to T_j$ be the canonical projection from the cusp U_j to T_j. Then there exists a form $\alpha_j\in\Omega^{0,1}(T_j;E_n)$ representing a non-trivial element in $\mathrm{H}^{0,1}(T_j;E_n)$ such that $\pi_j^*(\alpha_j)\in\Omega^1(U_j;E_n)$ is L^2.*

Proof. Let us work in the model of the half-space $\mathbf{H}^3 = \mathbf{C} \times (0, \infty)$. If $(z, t) = (x, y, t) \in \mathbf{H}^3$, the metric is given by

$$g = \frac{1}{t^2}(dx^2 + dy^2 + dt^2).$$

Proceeding as in the proof of Proposition 2.11, we obtain the form $\alpha = d\bar{z} \otimes X^{n-1}$. We will be done if we prove that $\pi_j^*(\alpha)$ is L^2. To compute the norm of $d\bar{z} \otimes X^{n-1}$, we may assume that the cusp U_j is isometric to $\mathbf{C} \times [1, \infty)/(\mathrm{Hol}_M \pi_1 T^2)$. Therefore, we have,

$$|d\bar{z} \otimes X^{n-1}|_{(w,t)} = |d\bar{z}|_{(w,t)} |X^{n-1}|_{(w,t)}.$$

On one hand,

$$|d\bar{z}|^2_{(w,t)} = |dx|^2_{(w,t)} + |dy|^2_{(w,t)} = 2t^2.$$

On the other hand, by definition of the metric of E_n, it can checked that

$$|X^{n-1}|^2_{(w,t)} = t^{1-n}|X^{n-1}|^2,$$

where $|X^{n-1}|$ is the norm of X^{n-1} in V_n respect to the fixed hermitian metric. Therefore, if R is a fundamental domain for T^2, we get

$$\int_{U_j} |d\bar{z} \otimes X^{n-1}|^2 d\,\mathrm{Vol}_{U_j} = 2|X^{n-1}|^2 \int_{R\times[1,\infty]} \frac{t^{3-n}}{t^3} dxdydt = C \int_1^\infty t^{-n} dt < \infty,$$

and the lemma is proved. ∎

Proposition 2.13. *Assume that $T_1, \ldots, T_r$ are all the connected components of $\partial\overline{M}$ such that $\mathrm{H}^0(T_j; E_n) \neq 0$. Then the following decomposition holds:*

$$\bigoplus_{j=1}^r \mathrm{H}^1(T_j; E_n) = \mathrm{Im}\, i^* \bigoplus_{j=1}^r \mathrm{H}^{0,1}(T_j; E_n).$$

Proof. It is enough to prove that $\mathrm{Im}\, i^* \cap \bigoplus_{j=1}^r \mathrm{H}^{0,1}(T_j; E_n) = 0$. Let $[\omega] \in \mathrm{H}^1(M; E_n)$ such that $i^*([\omega]) \in \bigoplus_{j=1}^k \mathrm{H}^{0,1}(T_j^2; E_n)$. Let us work with the cusps $U_j \cong T_j \times (0, \infty)$, and assume that they are disjoint. Let α_j be the forms given by the above lemma. Then

$$\omega = \lambda_j \pi_i^*(\alpha_j) + df_j, \quad \text{on } U_j,$$

for some $\lambda_j \in \mathbf{C}$ and $f_j \in \mathcal{C}^\infty(U_j; E_n)$. Let $F \in \mathcal{C}^\infty(M; E_n)$ such that $F_{|T_j \times [1,\infty)} = f_j$, and vanishes outside the cusps. By the above lemma, $\omega - dF$ is L^2, and hence the class $[\omega]$ has an L^2 representative, what implies that $[\omega] = 0$, as we wanted to prove. ∎

Proof of Proposition 2.10. Let α_j and β_j be generators of $\mathrm{H}^{0,1}(T_j^2; E_n)$ and $\mathrm{H}^{1,0}(T_j^2; E_n)$ respectively. We claim that the Kronecker pairing $([w_j \otimes \theta_j], \alpha_k)$ is zero for all j, k, and $([w_j \otimes \theta_j], \beta_k)$ is zero if and only if $j \neq k$. The we can assume that $k = j$. Let us fix T_j. Proceeding as in the proof of Proposition 2.11, we may assume that $w_j = X^{n-1}$, $\alpha_j = d\bar{z} \otimes X^{n-1}$ and $\beta_j = dz \otimes (zX + Y)^{n-1}$. We have

$$([w_j \otimes \theta_j], [\beta_j]) = \int_\theta \phi(X^{n-1}, (zX + Y)^{n-1}) dz = \int_\theta dz \neq 0. \tag{3}$$

On the other hand, $\phi(X^{n-1}, X^{n-1}) = 0$, and it follows that $([w_j \otimes \theta], [\alpha_j]) = 0$. This proves the claim.

Let us prove now the first assertion. Assume that

$$\sum_{j=1}^{r} \lambda_j i^* [w_j \otimes \theta_j] = 0, \quad \text{with } \lambda_j \in \mathbf{C}.$$

The naturality and the non-degeneracy of the Kronecker pairing imply that this is equivalent to

$$\sum_{j=1}^{r} \lambda_j \left([w_j \otimes \theta_j], i_*(\omega)\right) = 0, \quad \text{for all } [\omega] \in \mathrm{H}^1(M; E_n).$$

By Proposition 2.13, each β_j is uniquely written as

$$\beta_j = \gamma_j + \sum_{k=1}^{r} \mu_j^k \alpha_k, \quad \text{with } \gamma_j \in \operatorname{Im} i^* \text{ and } \mu_j^k \in \mathbf{C}.$$

Moreover, $(\gamma_1, \ldots, \gamma_r)$ is a basis of $\operatorname{Im} i^*$. The preceding discussion then implies $\lambda_j = 0$ for all j. The first assertion is thus proved.

Let us prove Assertion 2. The long exact sequence in homology for the pair $(\overline{M}, \partial \overline{M})$ shows that the inclusion $\partial \overline{M} \subset \overline{M}$ yields an isomorphism

$$i_* \colon \mathrm{H}_2(\partial \overline{M}; E_n) = \bigoplus_{j=1}^{r} \mathrm{H}_2(T_j; E_n) \to \mathrm{H}_2(M; E_n).$$

Thus it is enough to prove that $[w_j \otimes T_j]$ is not zero. This can be proved using Poincaré duality PD. Indeed, if we identify $\mathrm{H}^0(T_j; E_n)$ with the space invariant vectors, then it can be checked that

$$\mathrm{PD}(w_j) = [w_j \otimes T_j].$$

Assertion 3 follows easily from Equation 3. $\square$

3 Behaviour under hyperbolic Dehn surgery

The aim of this section is to analyse the behaviour of the n-th Reidemeister torsion under hyperbolic Dehn surgery. Let us fix some notation before discussing it.

Throughout this section M will denote an oriented, complete, hyperbolic 3-manifold of finite volume with l cusps. For each connected boundary component T_i of M we fix two closed simple oriented curves a_i, b_i in T_i generating $\mathrm{H}_1(T_i; \mathbf{Z})$. We define the following sets,

$$\begin{aligned} \mathcal{A} &= \{(p,q) = (p_1, \ldots, p_l, q_1, \ldots, q_l) \in \mathbf{Z}^l \times \mathbf{Z}^l \mid \gcd(p_i, q_i) = 1\}, \\ \mathcal{A}_M &= \{(p,q) \in \mathcal{A} \mid M_{p/q} := M_{p_1/q_1, \ldots, p_l/q_l} \text{ is hyperbolic } \}. \end{aligned}$$

Remark. We may regard $\mathcal{A}$ as a directed set respect to the following preorder

$$(p,q) \leq (p', q') \Leftrightarrow (p_i)^2 + (q_i)^2 \leq (p'_i)^2 + (q'_i)^2 \text{ for all } i = 1, \ldots, l.$$

The hyperbolic Dehn surgery theorem by Thurston implies that $\mathcal{A}_M$ is also directed set respect to the above preorder, namely any two elements of $\mathcal{A}_M$ have a common greater element. For the limit of an $\mathcal{A}_M$-net $\{x_{p/q}\}$ in some topological space, when it exists, we will use the notation

$$\lim_{(p,q) \to \infty} x_{p/q}.$$

In analysing the relation between the n-th dimensional torsion invariants of M with those of $M_{p/q}$, some issues arise. In order to discuss them, we distinguish two cases according to the parity of n.

We consider first the case $n = 2k+1$, with $k > 0$. In that case we find two difficulties. The first one is that we need some extra data in order to define the torsion invariant for M (we must choose non-trivial cycles $\theta_i \in \mathrm{H}_i(T_i; \mathbf{Z})$), whereas for $M_{p/q}$ this is already defined. The second one is due to the following result proved in [Por97, p. 110] (recall that our torsion is the inverse of the one considered in [Por97]),

$$\lim_{(p,q) \to \infty} \left| \tau_3(M_{p/q}) \right| = 0.$$

The proof of the above limit also works for any odd number $n \geq 3$. Moreover, the asymptotic growth of these sequences does not depend on the dimension n. These facts suggest that the above question should be formulated in terms of normalized torsions. In that case, we have the following result.

Proposition 3.1. *The accumulation point set of*

$$\left\{ \mathcal{T}_{2k+1}(M_{p/q}) \mid (p,q) \in \mathcal{A}_M \right\} \subset \mathbf{C}/\{\pm 1\}$$

is the segment joining the origin and the point $2^{2(k-1)l} \mathcal{T}_{2k+1}(M)$ *in* $\mathbf{C}/\{\pm 1\}$.

Let us analyse now the even dimensional case $n = 2k$, for $k > 0$. In this case, the main difficulty comes from the fact that we need a spin structure to define the n-th dimensional torsion invariant, so somehow we need a way to relate spin structures of M with those of $M_{p/q}$. To that end, for a fixed spin structure η on M, we define the following set,

$$\mathcal{A}_{M,\eta} = \left\{(p,q) \in \mathcal{A}_M \mid \eta \text{ can be extended to } M_{p/q}\right\}.$$

Remark. Notice that if η can be extended to $M_{p/q}$ then the extension is unique. In such case the extension will be denoted by $\eta_{p/q}$.

Using Corollary 2.8, we easily get the following characterization of $\mathcal{A}_{M,\eta}$.

Proposition 3.2. *For each T_i let $\epsilon_{a_i}, \epsilon_{b_i} = \pm 1$ be the sign of the trace of* $\mathrm{Hol}_{(M,\eta)}(a_i)$ *and* $\mathrm{Hol}_{(M,\eta)}(b_i)$ *respectively. Then $(p,q) \in \mathcal{A}_{M,\eta}$ if and only if*

$$\epsilon_{a_i}^{p_i}\epsilon_{b_i}^{q_i} = -1 \quad \text{for all } i = 1, \ldots, l.$$

Definition. We will say that a spin structure η on M is *compactly isolated* if $\mathcal{A}_{M,\eta}$ is empty; otherwise, we will say that η is *compactly approximable*.

Remark. If η is *compactly approximable*, Proposition 3.2 implies that $\mathcal{A}_{M,\eta}$ is infinite; in particular, $\mathcal{A}_{M,\eta}$ is a directed set as well. The terminology introduced in the above definition is coherent with the geometric topology of the space $\mathcal{MS}$ of spin hyperbolic 3-manifolds, see Section 4. For instance, if η is compactly approximable then the net of compact spin hyperbolic manifolds $\{(M_{p/q}, \eta_{p/q})\}_{(p,q)\in\mathcal{A}_{M,\eta}}$ converges to (M,η) in $\mathcal{MS}$.

As a corollary of the above proposition and the definition of an acyclic spin structure, we get the following result.

Corollary 3.3. *A spin structure η of M is compactly approximable if and only if it is acyclic.*

If η is compactly approximable, then $\mathrm{H}^*(M;\rho_{2k}) = 0$ for all $k > 0$, and hence it makes sense to consider the Reidemeister torsion $\tau(M;\rho_{2k})$. On the other hand, for all $(p,q) \in \mathcal{A}_{M,\eta}$ the spin structure $\eta_{p/q}$ of $M_{p/q}$ yields the spin hyperbolic representation

$$\mathrm{Hol}_{(M_{p/q},\eta_{p/q})} \colon \pi_1 M_{p/q} \to \mathrm{SL}(2,\mathbf{C}).$$

Composing this representation with V_{2k} we get a representation

$$\rho_{2k}^{p/q} \colon \pi_1 M_{p/q} \to \mathrm{SL}(2k,\mathbf{C}).$$

The compactness of $M_{p/q}$ guarantees the acyclicity of this representation. Hence, it makes sense to consider $\tau(M_{p/q};\rho_{2k}^{p/q})$.

Proposition 3.4. *Let η be a compactly approximable (equivalently acyclic) spin structure on M. Then the accumulation point set of*

$$\{\pm\,\tau(M_{p/q};\rho_{2k}^{p/q}) \mid (p,q)\in\mathcal{A}_{M,\eta}\}\subset \mathbf{C}/\{\pm 1\}$$

is the segment joining the origin and $\pm 2^{2kl}\,\tau(M;\rho_{2k})$.

The proof of both propositions will be based on surgery formulas for the torsions, which will be deduced from the Mayer-Vietoris formula for the Reidemeister torsion. These formulas involve the spin complex lengths of the core geodesics added on the Dehn filling. The above results then will follow essentially from the fact that the imaginary parts of the spin complex lengths of the added core geodesics in $M_{p/q}$ define a dense subset of $\mathbf{R}/\langle 4\pi\rangle$ as (p,q) varies in $\mathcal{A}_{M,\eta}$, see [Mey86].

3.1 Lifts and deformations

Consider a family of continuous local deformations of the complete structure,

$$\mathrm{Hol}_M\colon U\times\pi_1 M\to \mathrm{PSL}(2,\mathbf{C}),\quad u=(u_1,\dots,u_l)\in U\subset\mathbf{C}^l,$$

with U an open ball containing the origin, and with $\mathrm{Hol}_M(0,\cdot)=\mathrm{Hol}_M$. The open set U is usually called Thurston's slice, and is a double branched covering of a neighborhood of the variety of characters of M around the complete hyperbolic structure. We can assume that

$$\mathrm{Hol}_M(u,a_i)\sim\left[\begin{pmatrix} e^{u_i/2} & 1\\ 0 & e^{-u_i/2}\end{pmatrix}\right],\quad \mathrm{Hol}_M(u,b_i)\sim\left[\begin{pmatrix} e^{v_i(u)/2} & \tau_i(u)\\ 0 & e^{-v_i(u)/2}\end{pmatrix}\right],$$

where $v_i(u)$ and $\tau_i(u)$ are analytic functions on u which are related by

$$\sinh\frac{v_i(u)}{2}=\tau_i(u)\sinh\frac{u_i}{2}.$$

This last equation follows by imposing that the two matrices commute.

By Thurston's surgery theorem, for (p,q) large enough, the holonomy representation of the complete hyperbolic structure of $M_{p/q}$ is given at some value of u, say $u^{p/q}$, that is,

$$\mathrm{Hol}_M(u^{p/q},\gamma)=(\mathrm{Hol}_{M_{p/q}}\circ i_*)(\gamma),\quad \text{for all } \gamma\in\pi_1 M$$

where i_* is the induced morphism on the fundamental groups by the inclusion $i\colon M\hookrightarrow M_{p/q}$. Furthermore, we have

$$p_i u_i^{p/q}+q_i v_i(u^{p/q})=2\pi i.$$

Moreover, $u^{p/q}$ approaches to 0 as (p,q) goes to infinity.

For a fixed spin structure η on M, consider the lift of the whole family of representations $\mathrm{Hol}_M(u,\cdot)$ starting at $u=0$ with $\mathrm{Hol}_{(M,\eta)}$. By continuity, all these lifts are also group morphisms. Thus we obtain a family of representations

$$\mathrm{Hol}_{(M,\eta)}\colon U\times\pi_1 M\to \mathrm{SL}(2,\mathbf{C}).$$

The map $i_*\colon \pi_1 M\to\pi_1 M_{p/q}$ is surjective with kernel the subgroup generated by the curves $\{a_i^{p_i}b_i^{q_i}\}$ (here we are identifying $\mathrm{H}_1(T_i;\mathbf{Z})$ with $\pi_1 T_i$, and the latter group with a subgroup of $\pi_1 M$). Therefore, $\mathrm{Hol}_{(M,\eta)}$ yields a representation of $\pi_1 M_{p/q}$ if and only if

$$\mathrm{Hol}_{(M,\eta)}(u_{p/q},a_i^{p_i}b_i^{q_i})=\mathrm{Id},\quad \text{for all } i.$$

We can characterize this condition in terms of spin structures.

Lemma 3.5. *The representation* $\mathrm{Hol}_{(M,\eta)}(u^{p/q},\cdot)$ *of* $\pi_1 M$ *induces a representation of* $\pi_1 M_{p/q}$ *if and only if* $(p,q)\in\mathcal{A}_{M,\eta}$.

Proof. By Proposition 3.2, $(p,q)\in\mathcal{A}_{M,\eta}$ if and only if

$$\epsilon_{a_i}^{p_i}\epsilon_{b_i}^{q_i}=-1\quad \text{for all } i=1,\dots,l,$$

where $\epsilon_{a_i},\epsilon_{b_i}=\pm1$ is the sign of the trace of $\mathrm{Hol}_{(M,\eta)}(a_i)$ and $\mathrm{Hol}_{(M,\eta)}(b_i)$ respectively. On the other hand, for a fixed i, we can assume that

$$\mathrm{Hol}_{(M,\eta)}(u,a_i)=\epsilon_{a_i}\begin{pmatrix} e^{u_i/2} & 1\\ 0 & e^{-u_i/2}\end{pmatrix},\quad \mathrm{Hol}_M(u,b_i)=\epsilon_{b_i}\begin{pmatrix} e^{v_i(u)/2} & \tau_i(u)\\ 0 & e^{-v_i(u)/2}\end{pmatrix}.$$

Thus,

$$\left(\mathrm{Hol}_{(M,\eta)}(u^{p/q},a_i)\right)^{p_i}\left(\mathrm{Hol}_{(M,\eta)}(u^{p/q},b_i)\right)^{q_i}=-\epsilon_{a_i}^{p_i}\epsilon_{b_i}^{q_i},$$

where we have used the equation $p_i u_i^{p/q}+q_i v_i(u^{p/q})=2\pi i$. The result then follows immediately. $\square$

3.2 Even dimensional case

Let η be a compactly approximable (equivalently, acyclic) spin structure on M. Consider the family of representations $\mathrm{Hol}_{(M,\eta)}(u,\cdot)$ introduced in the preceding section. Composing it with the $2k$-dimensional irreducible representation of $\mathrm{SL}(2,\mathbf{C})$, we get a family of representations

$$\rho_{2k}\colon U\times\pi_1 M\to\mathrm{SL}(2k,\mathbf{C}).$$

Since η is acyclic, for $u=0$ the representation $\rho_{2k}(u,\cdot)$ is acyclic. The following more or less well known result then implies that $\rho_{2k}(u,\cdot)$ is also acyclic for u close to 0.

Proposition 3.6. *Let X be a finite CW-complex, and consider a continuous family of representations*

$$\rho\colon U \times \pi_1(X, x_0) \to \mathrm{GL}(n, \mathbf{C}),$$

where U is some space of parameters. For a fixed $m \geq 0$, define the map $F\colon U \to \mathbf{Z}$ by $F(u) = \dim \mathrm{H}_m(X; \rho_u)$, where $\rho_u := \rho(u, \cdot)$. Then F is upper semicontinuous, that is,

$$\limsup_{u \to u_0} F(u) \leq F(u_0).$$

Proof. The idea is that the rank of a matrix, viewed as a map on the space of matrices, is lower a semicontinuous function. The details are as follows. The homology groups $\mathrm{H}_*(X; \rho_u)$ can be defined as the homology groups of the complex

$$\left(V \otimes_{\rho(u)} C_*(\widetilde{X}; \mathbf{Z}), \mathrm{Id} \otimes \partial_*\right).$$

Let us fix $(w_1, \ldots, w_n)$ a basis of V. Let $\{e^j_1, \ldots, e^j_{i_j}\}$ be the cells of X of dimension j, and let $\{\tilde{e}^j_1, \ldots, \tilde{e}^j_{i_j}\}$ be fixed lifts of these cells to $\widetilde{X}$. Then the set $\{w_i \otimes \tilde{e}^j_k\}$ gives a basis of $V \otimes_{\rho_u} C_j(\widetilde{X}; \mathbf{Z})$. Respect to these basis, the boundary map $\partial_j(u)$ is written as a matrix $A_j(u)$ whose entries depend continuously on u. Then we have

$$F(u) = \dim \mathrm{Ker}\, A_j(u) - \mathrm{rank}\, A_{j+1}(u).$$

Since the rank of a matrix is lower semicontinuous, the dimension of the kernel is upper-semicontinuous, and hence $F(u)$ is upper semicontinuous. $\square$

Remark. The above result can be seen as a special case of a further result stated in [Har77] as “The semicontinuity Theorem”, which establishes the upper semicontinuity of the dimension function of some cohomology groups in a much more general context.

Let us put $\rho_{2k}(u) := \rho_{2k}(u, \cdot)$. The above proposition shows that it makes sense to consider $\tau(M; \rho_{2k}(u))$ for u close to 0. On the other hand, for $(p, q) \in \mathcal{A}_{M,\eta}$ large enough, the representation $\rho_{2k}(u^{p/q})$ induces a representation

$$\rho^{p/q}_{2k}\colon \pi_1 M_{p/q} \to \mathrm{SL}(2k, \mathbf{C}),$$

such that $\rho^{p/q}_{2k} \circ i_* = \rho_{2k}(u_{p/q})$. Since $M_{p/q}$ is compact, the representation $\rho^{p/q}_{2k}$ is acyclic. Therefore, it also makes sense to consider $\tau(M_{p/q}; \rho^{p/q}_{2k})$. The following lemma gives the relationship between these two quantities.

Lemma 3.7. *Let $\gamma_1, \ldots, \gamma_l$ be the core geodesics added on the (p, q)-Dehn filling $M_{p/q}$, and $\lambda_{p/q}$ be the spin complex length function respect to the spin hyperbolic structure $\eta_{p/q}$. Then we have*

$$\tau(M_{p/q}; \rho^{p/q}_{2k}) = \pm\, \tau(M; \rho_{2k}(u_{p/q})) \prod_{j=0}^{k-1} \prod_{i=1}^{l} \left(e^{(\frac{1}{2}+j)\lambda_{p/q}(\gamma_i)} - 1\right)\left(e^{-(\frac{1}{2}+j)\lambda_{p/q}(\gamma_i)} - 1\right).$$

Proof. By induction, we can assume that M has one cusp. We will apply the Mayer-Vietoris sequence to the decomposition $M_{p/q} = M \cup N(\gamma)$, where $N(\gamma)$ is a tubular neighbourhood of the core geodesic added on the Dehn filling. We must show first that all involved spaces are $\rho_{2k}^{p/q}$-acyclic. We have already seen it for M. Since $\mathrm{Hol}_{M_{p/q}}(\gamma)$ has no fixed vector other that 0, $\mathrm{H}^0(\gamma; \rho_{2k}^{p/q})$ is trivial, and hence so is $\mathrm{H}^1(\gamma; \rho_{2k}^{p/q})$; that proves that $N(\gamma) \simeq \gamma$ is acyclic. The same argument shows that $\mathrm{H}^r(\partial N(\gamma)); \rho_{2k}^{p/q})$ is trivial for $r = 0, 2$, what implies (Euler characteristic argument) that so is for $r = 1$. The Mayer-Vietoris sequence then yields the formula,

$$\tau(M_{p/q}; \rho_{2k}^{p/q})\, \tau(\partial N(\gamma); \rho_{2k}^{p/q}) = \tau(M; \rho_{2k}^{p/q})\, \tau(\gamma; \rho_{2k}^{p/q}).$$

The torsion of the torus $\partial N(\gamma)$ is ± 1, as it is the Reidemeister torsion of an even dimensional manifold, see [Mil62]. Finally, an easy computation shows that

$$\tau(\gamma; \rho_{2k}^{p/q}) = \prod_{j=0}^{k-1} \big(e^{(\frac{1}{2}+j)\lambda_{p/q}(\gamma)} - 1\big)\big(e^{-(\frac{1}{2}+j)\lambda_{p/q}(\gamma)} - 1\big).$$

□

Now we can prove Proposition 3.4.

Proof of Propostion 3.4 . The above lemma can be written as

$$\frac{\tau(M_{p/q}; \rho_{2k}^{p/q})}{\tau(M; \rho_{2k}(u_{p/q}))} = 2^{2kl} \prod_{j=0}^{k-1} \prod_{i=1}^{l} \frac{1 - \cosh\big((\frac{1}{2} + j)\lambda_{p/q}(\gamma_i)\big)}{2}.$$

Since $\tau(M; \rho_{2k}(u_{p/q}))$ converges to $\tau(M; \rho_{2k})$ as (p, q) goes to infinity (this can be proved in the same way as 3.6), it remains to prove that the product of the right hand side of the above equation is dense in $[0, 1]$. To prove this, consider the map defined by

$$\begin{aligned} F\colon [0,\infty) \times [0, 4\pi] &\longrightarrow \mathbf{C} \\ (t, \theta) &\longmapsto \prod_{j=0}^{k-1} \frac{1 - \cosh\big((\frac{1}{2} + j)(t + \theta\, \mathrm{i})\big)}{2}. \end{aligned}$$

The image of $\{0\} \times [0, 4\pi]$ under F is $[0, 1]$, since $F(\{0\} \times [0, 4\pi]) \subset [0, 1]$, $F(0, 0) = 0$ and $F(0, 2\pi) = 1$. The result then follows from the fact that the real part of $\lambda_{p/q}(\gamma_i)$ goes to zero, and its imaginary part is dense in $[0, 4\pi]$, see [Mey86]. □

3.3 Odd dimensional case

We will use the same notation as in the previous subsections. Throughout this subsection we will assume that $n = 2k + 1$ and $k > 0$.

Lemma 3.8. *Let T_j be a fixed boundary component of $\partial\overline{M}$. Assume that*

$$\mathrm{Hol}_M(u,a_j) = \left[\begin{pmatrix} e^{u_j/2} & 1 \\ 0 & e^{-u_j/2}\end{pmatrix}\right], \quad \mathrm{Hol}_M(u,b_j) = \left[\begin{pmatrix} e^{v_j(u)/2} & \tau_j(u) \\ 0 & e^{-v_j(u)/2}\end{pmatrix}\right],$$

where a_j, b_j are generators of the fundamental group of T_j. For $u_j \neq 0$ consider the following vector

$$w_j(u) := X^k \left(X - 2\sinh\frac{u_j}{2} Y\right)^k \in V_{2k+1} \cong S_{2k}[X,Y],$$

where $S_n[X,Y]$ is the space of homogeneous polynomials of degree n in the variables X, Y. Then, for $u_j \neq 0$ close to 0, the vector $w_j(u)$ is $\rho_{2k+1}(u)$-invariant. Moreover, the map

$$\begin{array}{ccc} \Omega^*(T_j;\mathbf{C}) & \to & \Omega^*(T_j;\rho_{2k+1}(u)) \\ \omega & \mapsto & \omega\otimes w_j(u) \end{array}$$

induces isomorphisms in cohomology.

Proof. Let $\mathrm{Hol}_{(M,\eta)}$ be a lift of the holonomy representation. Since both $\mathrm{Hol}_{(M,\eta)}(a_j)$ and $\mathrm{Hol}_{(M,\eta)}(b_j)$ diagonalize and commute, there exists a basis (e_1,e_2) of $\mathbf{C}^2$ that simultaneously diagonalize them. We can take $e_1 = X$, and $e_2 = X - 2\sinh\frac{u_j}{2}Y$. The vector $e_1^k e_2^k \in V_{2k+1}$ is then independent on the chosen lift, and invariant by both $\mathrm{Hol}_{(M,\eta)}(a_j)$ and $\mathrm{Hol}_{(M,\eta)}(b_j)$. This shows that $w_j(u)$ is $\rho_{2k+1}(u)$-invariant, and the first part of the lemma is proved.

The vector $w_j(u)$ gives a parallel nowhere vanishing section of the flat vector bundle $E_n(u)$ defined by $\rho_n(u)$. On the other hand, the $\mathrm{SL}(2,\mathbf{C})$-invariant pairing

$$\phi\colon V_n\times V_n\to\mathbf{C},$$

defines a non-degenerate symmetric bilinear form on $E_n(u)$. We have,

$$\phi\left(w_j(u),w_j(u)\right) = 2\left(-2\sinh\frac{u_j}{2}\right)^k.$$

Therefore, for $\sinh\frac{u_j}{2}\neq 0$, we have a decomposition $E_n|T_j = L\oplus L^\perp$, where L is the line bundle defined by $w_j(u)$, and $L^\perp$ is the orthogonal complement respect to φ. Note that both sub-bundles are flat, so we have

$$\mathrm{H}^*(T_j;E_n) = \mathrm{H}^*(T_j;L)\oplus\mathrm{H}^*(T_j;L^\perp).$$

Counting dimensions, we deduce that $\mathrm{H}^0(T_j;E_n) = \mathrm{H}^0(T_j;L)\cong\mathrm{H}^*(T_j;\mathbf{C})$, where last isomorphism is given by tensorizing by $w_j(u)$. This proves the last assertion of the lemma for degree 0. The lemma then follows by Poincaré duality and an Euler characteristic argument. $\square$

Proposition 3.9. *There exists a neighbourhood of the origin $W\subset U$ such that for all $u\in W$,*

$$\dim_{\mathbf{C}}\mathrm{H}_1(M;\rho_{2k+1}(u)) = \dim_{\mathbf{C}}\mathrm{H}_2(M;\rho_{2k+1}(u)) = l,$$

where l is the number of connected components of $\partial\overline{M}$.

Proof. By Poincaré duality and an Euler characteristic argument, we deduce that

$$\dim_{\mathbf{C}} \mathrm{H}_1(M;\rho_{2k+1}(u)) = \dim_{\mathbf{C}} \mathrm{H}_1(M,\partial M;\rho_{2k+1}(u)).$$

The long exact sequence of the pair $(\overline{M},\partial\overline{M})$ yields the following short exact sequence,

$$\mathrm{H}_1(\overline{M},\partial\overline{M};\rho_{2k+1}(u)) \to \mathrm{H}_0(\partial\overline{M};\rho_{2k+1}(u)) \to 0.$$

Therefore,

$$\dim_{\mathbf{C}} \mathrm{H}_1(M;\rho_{2k+1}(u)) \geq \dim_{\mathbf{C}} \mathrm{H}_0(\partial\overline{M};\rho_{2k+1}(u)) = \sum_{j=1}^{l} \dim_{\mathbf{C}} \mathrm{H}_0(T_j;\rho_{2k+1}(u)).$$

The vector space $\mathrm{H}_0(T_j;\rho_{2k+1}(u))$ has dimension 1. Indeed, if $u_j = 0$ this is clear by direct inspection, and if $u_j \neq 0$, this follows from Lemma 3.8. Hence,

$$\dim_{\mathbf{C}} \mathrm{H}_1(M;\rho_{2k+1}(u)) \geq l, \quad \text{for } u \in U.$$

Since $\dim_{\mathbf{C}} \mathrm{H}_1(M;\rho_{2k+1}(0)) = l$, the upper semicontinuity of the dimension function (Proposition 3.6) implies the result. ☐

Proposition 3.10. *Let $\{\theta_j\}$ be a collection of nontrivial cycles with $\theta_j \in \mathrm{H}_1(T_j;\mathbf{Z})$. Then there exists a neighbourhood of the origin $W \subset U$ such that for all $u \in W$ the following assertions hold:*

1. *A basis of $\mathrm{H}_1(M;\rho_{2k+1}(u))$ is given by*

$$\left(i_*[w_1(u)\otimes\theta_1],\ldots,i_*[w_l(u)\otimes\theta_l]\right).$$

2. *A basis of $\mathrm{H}_2(M;\rho_{2k+1}(u))$ is given by*

$$\left(i_*[w_1(u)\otimes T_1],\ldots,i_*[w_l(u)\otimes T_l]\right).$$

In both cases, the vectors $w_j(u)$ are the ones given by Lemma 3.8, $[T_j] \in \mathrm{H}_2(T_j;\mathbf{Z})$ is a fundamental class of ∂M, and i_* is the map induced in homology by the inclusion $i\colon \partial\overline{M} \to \overline{M}$.

Proof. Proposition 2.10 shows that the two assertions are true for $u = 0$. The result then follows proceeding as in the proof of Proposition 3.6. ☐

Therefore, it makes sense to consider $\tau(M;\rho_{2k+1}(u_{p/q});\{\theta_j\})$, the Reidemeister torsion of M respect to the (non-acyclic) representation $\rho_{2k+1}(u_{p/q})$ and the bases in homology given by the above proposition associated to the family of non-trivial cycles $\{\theta_j\}$. We would like to get a surgery formula for $\tau(M;\rho_{2k+1}(u_{p/q});\{\theta_j\})$. It turns out that it is easier to work with the bases in homology given by the following lemma.

Lemma 3.11. *For sufficiently large (p,q), a basis of* $\mathrm{H}_1(M;\rho_{2k+1}(u_{p/q}))$ *is given by,*

$$\left(i_*[w_1(u_{p/q})\otimes(p_1a_1+q_1b_1)],\dots,i_*[w_l(u_{p/q})\otimes(p_la_l+q_lb_l)]\right).$$

Proof. This is a Mayer-Vietoris argument as in Lemma 3.7. We have the decomposition

$$M_{p/q}=M\cup N,\quad \text{with } N=\bigcup_{j=1}^{l}N(\gamma_j),$$

where $\{N(\gamma_j)\}$ is a collection of disjoint tubular neighbourhoods of the core geodesics γ_j added in the Dehn filling. By compactness, $M_{p/q}$ is $\rho_n(u_{p/q})$-acyclic. The Mayer-Vietoris exact sequence then gives an isomorphism

$$\mathrm{H}_*(\partial M;\rho_n(u_{p/q}))\cong \mathrm{H}_*(M;\rho_n(u_{p/q}))\oplus\mathrm{H}_*(N;\rho_n(u_{p/q})).$$

We have that $\mathrm{H}_*(T_j;\mathbf{C})$ and $\mathrm{H}_*(T_j;\rho_n(u_{p/q}))$ are isomorphic via tensorization $w_j(u_{p/q})\otimes -$ (this is the homological counterpart of lemma 3.8). The same isomorphisms also holds true for $N(\gamma_j)\simeq\gamma_j$. Since $[p_ja_j+q_jb_j]\in\mathrm{H}_1(N(\gamma_j);\mathbf{Z})$ is zero by construction, the described vectors must be linearly independent. ☐

Lemma 3.12. *Let $\gamma_1,\dots,\gamma_l$ be the core geodesics added on the (p,q)-Dehn filling $M_{p/q}$, and $\lambda_{p/q}$ be the complex length function of $M_{p/q}$. Then we have*

$$\tau(M_{p/q};\rho_{2k+1}^{p/q})=\tau(M;\rho_{2k+1}(u_{p/q}),\{p_ja_j+q_jb_j\})\prod_{j=1}^{k}\prod_{i=1}^{l}(e^{j\lambda_{p/q}(\gamma_i)}-1)(e^{-j\lambda_{p/q}(\gamma_i)}-1).$$

Proof. This is again a Mayer-Vietoris argument. With the same notation as in the preceding proof, we have $M_{p/q}=M\cup N$. The formula for the torsion is

$$\tau(M_{p/q};\rho_{2k+1}^{p/q})\tau(\partial M;\rho_{2k+1}^{p/q})=\tau(M;\rho_{2k+1}^{p/q},\{p_ja_j+q_jb_j\})\tau(N;\rho_{2k+1}^{p/q})\tau(\mathcal{H}_*),$$

where $\tau(\mathcal{H}_*)$ is the torsion of the Mayer-Vietoris complex computed using the bases that has been chosen to compute the involved torsions in the decomposition. To compute the torsions we choose bases in homology as follows. For $\mathrm{H}_*(T_j;\rho_{2k+1}^{p/q})$, we take in degree 0, $[P_n^j(u_{p/q})\otimes\sigma_j]$, where σ_j is a generator of $\mathrm{H}_0(T_j;\mathbf{Z})$, in degree 1, $[P_n^j(u_{p/q})\otimes(p_ja_j+q_jb_j)]$, and in degree 2, $[P_n^j(u_{p/q})\otimes T_j]$. For $\mathrm{H}_*(N(\gamma_j);\rho_{2k+1}^{p/q})$, we take in degree 0, $[P_n^j(u_{p/q})\otimes i_{2,*}(\sigma_j)]$, and in degree 1, $[P_n^j(u_{p/q})\otimes i_{2,*}(\gamma)]$, where $i_{2,*}$ is the map induced by the inclusion $i_2\colon\partial M=\partial N\to N$, and $\gamma\in\mathrm{H}_1(\partial M;\mathbf{Z})$ is such that $i_*^1(\gamma)\in\mathrm{H}_1(M;\rho_{2k+1}(u_{p/q})$ is zero (notice that such a curve always exists and $i_{2,*}(\gamma)\in\mathrm{H}^1(D^2\times S^1;\mathbf{Z})$ is homologous to the core geodesic). Respect to these bases, we have $\tau(\mathcal{H}_*)=1$, since the isomorphism $i_{1,*}+i_{2,*}$ appearing in the Mayer-Vietoris

sequence is represented by the identity matrix. On the other hand, the torsion of ∂M is ± 1, as it is an even dimensional manifold. Thus we have

$$\tau(M_{p/q};\rho^{p/q}_{2k+1}) = \tau(M;\rho^{p/q}_{2k+1},\{p_j a_j + q_j b_j\}) \prod_{j=1}^{l} \tau(\gamma_j;\rho^{p/q}_{2k+1}).$$

Finally, a straightforward computation gives

$$\tau(\gamma_j;\rho^{p/q}_{2k+1}) = \prod_{h=1}^{k} (e^{h\lambda_{p/q}(\gamma_j)} - 1)(e^{-h\lambda_{p/q}(\gamma_j)} - 1).$$

□

If we "normalize torsions" in the formula given in the above lemma, then we get

$$\mathcal{T}_{2k+1}(M_{p/q}) = \frac{\tau(M;\rho_{2k+1}(u_{p/q}),\{p_j a_j + q_j b_j\})}{\tau(M;\rho_3(u_{p/q}),\{p_j a_j + q_j b_j\})} \prod_{j=2}^{k}\prod_{i=1}^{l} (e^{j\lambda_{p/q}(\gamma_i)} - 1)(e^{-j\lambda_{p/q}(\gamma_i)} - 1).$$

Let us focus on the quotient of torsions appearing in the right hand side of the above equation. We shall write down a formula relating the torsion of M respect to the basis $\{a_j\}$) and $\{p_j a_j + q_j b_j\}$), for sufficiently large (p,q). To that end, let $A_{2k+1}(p,q)$ be the change of basis matrix from the basis $\{[w_j(u_{p/q}) \otimes a_j]\}$ to $\{[w_j(u_{p/q}) \otimes (p_j a_j + q_j b_j)]\}$. Then the change of basis formula for the torsion yields,

$$\tau(M;\rho_{2k+1}(u_{p/q}),\{p_j a_j + q_j b_j\}) \det A_{2k+1}(p,q) = \tau(M;\rho_{2k+1}(u_{p/q}),\{a_j\}).$$

This equation implies

$$\frac{\tau(M;\rho_{2k+1}(u_{p/q}),\{p_j a_j + q_j b_j\})}{\tau(M;\rho_3(u_{p/q}),\{p_j a_j + q_j b_j\})} = \frac{\tau(M;\rho_{2k+1}(u_{p/q}),\{a_j\})}{\tau(M;\rho_3(u_{p/q}),\{a_j\})} \frac{\det A_3(p,q)}{\det A_{2k+1}(p,q)}. \tag{4}$$

Working as in in Proposition 3.6, we can prove that

$$\lim_{u\to 0} \tau(M;\rho_{2k+1}(u),\{a_j\}) = \tau(M;\rho_{2k+1}(0),\{a_j\}) = \tau(M;\rho_{2k+1},\{a_j\}).$$

Hence,

$$\lim_{(p,q)\to\infty} \frac{\tau(M;\rho_{2k+1}(u_{p/q}),\{p_j a_j + q_j b_j\})}{\tau(M;\rho_3(u_{p/q});\{p_j a_j + q_j b_j\})} = \mathcal{T}_{2k+1}(M) \lim_{(p,q)\to\infty} \frac{\det A_3(p,q)}{\det A_{2k+1}(p,q)}.$$

Lemma 3.13. *For any $k \geq 3$,*

$$\lim_{(p,q)\to\infty} \frac{\det A_{2k+1}(p,q)}{\det A_3(p,q)} = 1.$$

Proof. We have

$$A_{2k+1}(p,q) = \operatorname{diag}(p) + \operatorname{diag}(q) B_{2k+1}(u_{p/q}),$$

where $B_{2k+1}(u)$ is the change of basis matrix from the basis $\{[w_j(u) \otimes a_j]\}$ to $\{[w_j(u) \otimes b_j]\}$. Working as in Proposition 3.6, it can be checked that $B_{2k+1}(u)$ depends analytically on u. Note that at $u = 0$ we have

$$B_{2k+1}(0) = \operatorname{diag}(\operatorname{cshape}(b_1, a_1), \ldots, \operatorname{cshape}(b_l, a_l)).$$

Let us write $P = \operatorname{diag}(p)$, $Q = \operatorname{diag}(q)$ and $C = B_{2k+1}(0)$. Notice that C is independent of k. The lemma will follow easily once we prove the following equality

$$\lim_{(p,q)\to\infty} \frac{\det(P + QC)}{\det(P + QB_{2k+1}(u_{p/q}))} = 1.$$

We have,

$$\frac{\det(P + QC)}{\det(P + QB_{2k+1}(u_{p/q}))} = \frac{\det(Q^{-1}P + C)}{\det(Q^{-1}P + B_{2k+1}(u_{p/q}))}$$

Let us put $D = Q^{-1}P + C$ and $E(u_{p/q}) = B_{2k+1}(u_{p/q}) - C$. Then we have

$$\begin{aligned} \frac{\det(P + QC)}{\det(P + QB_{2k+1}(u_{p/q}))} &= \frac{\det D}{\det(D + E_{2k+1}(u_{p/q}))} \\ &= \frac{1}{\det(\operatorname{Id} + D^{-1}E_{2k+1}(u_{p/q}))}. \end{aligned}$$

If $D = (d_{ij})$ then we have

$$|d_{jj}| = |p_j/q_j + \operatorname{cshape}(a_j, b_j)| > |\operatorname{Im} \operatorname{cshape}(a_j, b_j)| > 0.$$

Therefore, the entries of the diagonal matrix D^{-1} are bounded, and hence

$$\lim_{(p,q)\to\infty} D^{-1}E_{2k+1}(u_{p/q}) = \lim_{(p,q)\to\infty} D^{-1}(B_{2k+1}(u_{p/q}) - B_{2k+1}(0)) = 0.$$

$\square$

Therefore, if we take take limits in equation 4, we get

$$\lim_{(p,q)\to\infty} \frac{\tau(M; \rho_{2k+1}(u_{p/q}), \{p_j a_j + q_j b_j\})}{\tau(M; \rho_3(u_{p/q}); \{p_j a_j + q_j b_j\})} = \frac{\tau(M; \rho_{2k+1}, \{a_j\})}{\tau(M; \rho_3, \{a_j\})} = \mathcal{T}_{2k+1}(M).$$

Just for future references, let us summarize the preceding results in the following lemma.

Lemma 3.14. *With the above notation, for $k>1$ we have*

$$\mathcal{T}_{2k+1}(M_{p/q}) = \frac{\det A_3(p,q)}{\det A_{2k+1}(p,q)} \frac{\tau(M;\rho_{2k+1}(u_{p/q}),\{a_j\})}{\tau(M;\rho_3(u_{p/q}),\{a_j\})} \prod_{j=2}^{k}\prod_{i=1}^{l}(e^{j\lambda_{p/q}(\gamma_i)}-1)(e^{-j\lambda_{p/q}(\gamma_i)}-1).$$

Moreover,

$$\begin{aligned} \lim_{(p,q)\to\infty} \frac{\det A_{2k+1}(p,q)}{\det A_3(p,q)} &= 1, \\ \lim_{(p,q)\to\infty} \frac{\tau(M;\rho_{2k+1}(u_{p/q}),\{a_j\})}{\tau(M;\rho_3(u_{p/q}),\{a_j\})} &= \mathcal{T}_{2k+1}(M). \end{aligned}$$

Proof of Proposition 3.1. By the above lemma 3.14, the result is reduced to prove that the accumulation point set of

$$\left\{ \prod_{j=2}^{k}\prod_{i=1}^{l}(e^{j\lambda_{p/q}(\gamma_i)}-1)(e^{-j\lambda_{p/q}(\gamma_i)}-1) \right\}_{(p,q)\in\mathcal{A}_M}$$

is $[0,4^{(k-1)l}]$, what may be proved as in the even dimensional case, Proposition 3.4. □

4 Continuity of the complex length spectrum

The aim of this section is to prove the continuity of the complex length spectrum in a sense that we shall precise in the subsequent subsections.

4.1 Closed geodesics in a hyperbolic manifold

Although the material of this subsection is well known, we think it is worth to review it for the sake of completeness.

Let M be an oriented complete hyperbolic 3-manifold, and Hol_M be its holonomy representation. Let us consider $\mathcal{C}(M)$ the set of closed (constant-speed) geodesics in M up to orientation preserving reparametrisation. We will describe $\mathcal{C}(M)$ as the following quotient set,

$$\mathcal{C}(M) = \left\{\varphi\colon S^1 \to M \mid \varphi \text{ is a geodesic}\right\}/S^1.$$

The action of S^1 on a closed geodesic is given by translation on the parameter. We are interpreting S^1 as $\mathbf{R}/\mathbf{Z}$. If $k\in\mathbf{Z}$ and $\varphi\colon S^1\to M$ is a closed geodesic, $k\varphi$ will denote the closed geodesic $t\mapsto\varphi(kt)$.

Definition. A closed geodesic φ is said to be *prime* if $\varphi\neq k\psi$ for any $k>1$ and any closed geodesic ψ (i.e. φ is prime if it traces its image exactly once). A class $[\varphi]\in\mathcal{C}(M)$ is said to be prime if φ is prime. The set of prime classes of $\mathcal{C}(M)$ will be denoted by $\mathcal{PC}(M)$.

We will also need the group theoretic definition of primality.

Definition. Let G be a group. An element $g \in G$ is said to be *prime* if $g \neq h^k$ for all $h \in G$ and $k > 1$ (note that we are excluding the identity from this definition). If $\mathrm{C}(G)$ denotes the set of conjugacy classes of G, then $[g] \in \mathrm{C}(G)$ is said to be prime if g is prime.

The identification between the set $\mathrm{C}(\pi_1(M,p))$ of conjugacy classes of $\pi_1(M,p)$ and loops in M up to free homotopy, yields a natural map

$$\psi \colon \mathcal{C}(M) \to \mathrm{C}(\pi_1(M,p)).$$

Let $\mathrm{HypC}(\pi_1(M,p))$ be the set of hyperbolic conjugacy classes of $\pi_1(M,p)$, that is,

$$\mathrm{HypC}(\pi_1(M,p)) = \{[\gamma] \in \mathrm{C}(\pi_1(M,p)) \mid \mathrm{Hol}_M(\gamma) \text{ is of hyperbolic type}\}.$$

The following result is well known, and is easily deduced from the fact that an isometry of $\mathbf{H}^3$ of hyperbolic type has exactly one axis.

Proposition 4.1. *The natural map $\psi \colon \mathcal{C}(M) \to \mathrm{C}(\pi_1(M,p))$ is a bijection onto the set $\mathrm{HypC}(\pi_1(M,p))$. Moreover, $[\varphi] \in \mathcal{C}(M)$ is prime if and only if so is $\psi([\varphi])$.*

It is useful to endow the set $\mathcal{C}(M)$ with the (quotient) supremum metric. More explicitly, if $[\varphi_1], [\varphi_2] \in \mathcal{C}(M)$ then its distance is defined by

$$d([\varphi_1],[\varphi_2]) = \min_{s\in S^1} \max_{t \in S^1} \{d(\varphi_1(t+s), \varphi_2(t))\}.$$

The following observation is an immediate consequence of the previous proposition, and will be used quite often in the subsequent subsections.

Proposition 4.2. *Let $[\varphi], [\varphi']$ be two distinct elements of $\mathcal{C}(M)$, and let m be the minimum of the injectivity radius at φ. Then $d([\varphi],[\varphi']) \geq m$.*

Proof. Assume that $d([\varphi],[\varphi']) = m' < m$. With suitable parametrisations, we have that for all $t \in S^1$, $d(\varphi(t), \varphi'(t)) \leq m'$. By the hypothesis on the injectivity radius, there exists a unique minimizing geodesic joining $\varphi(t)$ and $\varphi'(t)$. Therefore, we can define a free homotopy from φ to φ', what contradicts Proposition 4.1. $\square$

We want to end this subsection with an estimate on the growth of the number of closed geodesics in function of their length. The following estimation, though not the best possible (see for instance [Mar69], [CK02]), has the advantage of being explicit. Its proof is very close to the proof of Lemma 5.3 in [CK02].

Lemma 4.3. *Let M be a complete hyperbolic 3-manifold. For a compact domain $K \subset M$ define*

$$\mathcal{P}_K(t) = \#\left\{\varphi \in \mathcal{C}(M) \mid \varphi(S^1) \cap K \neq \emptyset, l(\varphi) \leq t\right\}.$$

Then, $\mathcal{P}_K(t) \leq Ce^{2t}$, with $C = \pi \frac{e^{8 \operatorname{diam} K}}{\operatorname{Vol} K}$.

Proof. Let $M = \mathbf{H}^3/\Gamma$, with Γ a subgroup of $\operatorname{Isom}\mathbf{H}^3$, and let $\pi\colon \mathbf{H}^3 \to M$ denote the covering projection. Pick a point $p \in \mathbf{H}^3$ with $\pi(p) \in K$ and consider the Dirichlet domain centred at p:

$$D(p) = \{x \in \mathbf{H}^3 \mid d(x, \gamma(p)) \leq d(x,p),\ \forall \gamma \in \Gamma\}.$$

The intersection $\tilde{K} = D(p) \cap \pi^{-1}(K)$ is a fundamental domain for K, which means that $\pi^{-1}(K) = \bigcup_{\gamma\in\Gamma} \gamma(\tilde{K})$ and $\operatorname{Vol}(\gamma_1(\tilde{K}) \cap \gamma_2(\tilde{K})) = 0$, $\forall \gamma_1 \neq \gamma_2 \in \Gamma$. Moreover $\operatorname{diam}\tilde{K} \leq 2\operatorname{diam} K$ and $\operatorname{Vol}\tilde{K} = \operatorname{Vol} K$.

Now let $\varphi \in \mathcal{C}(M)$ intersecting K. Then there exists an isometry $\gamma \in \Gamma$ of hyperbolic type representing φ whose axis intersects $\tilde{K}$. We claim that

$$\gamma(\tilde{K}) \subset B(p, 4\operatorname{diam} K + l(\varphi)).$$

To prove this inclusion, we pick a point $q \in \tilde{K}$ that lies in the axis of γ. For any $q' \in \tilde{K}$,

$$d(p, \gamma(q')) \leq d(p,q) + d(q, \gamma(q)) + d(\gamma(q), \gamma(q')) \leq 4\operatorname{diam} K + d(q, \gamma(q))$$

and $d(q,\gamma(q)) = l(\varphi)$, which proves the claim. Hence, for any geodesic contributing to $\mathcal{P}_K(t)$, there is a hyperbolic isometry whose axis is a lift of this geodesic and such that $\gamma(\tilde{K}) \subset B(p, 4\operatorname{diam} K + t)$. In addition, $\operatorname{Vol}(\gamma_1(\tilde{K}) \cap \gamma_2(\tilde{K})) = 0$, $\forall \gamma_1 \neq \gamma_2 \in \Gamma$. Thus we get the inequality:

$$\mathcal{P}_K(t)\operatorname{Vol} K = \mathcal{P}_K(t)\operatorname{Vol}\tilde{K} \leq \operatorname{Vol} B_p(4\operatorname{diam} K + t) \leq \pi e^{8\operatorname{diam} K + 2t}.$$

We have used that the volume of a ball of radius R in $\mathbf{H}^3$ is less than πe^{2R}. $\square$

4.2 Complex length spectrum

To any closed geodesic $\varphi \in \mathcal{C}(M)$, we can attach two geometric invariants: its length and its geometric torsion. Recall that the geometric torsion of φ is defined as the oriented angle formed between an orthogonal vector to φ and the parallel transport of it along φ.

In terms of the holonomy representation these two invariants are the translation distance and the rotational part of the corresponding hyperbolic isometry. More explicitly, if $[\gamma] \in \operatorname{HypC}(\pi_1(M,p))$, then

$$\operatorname{Hol}_M(\gamma) \sim \left[\begin{pmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{pmatrix}\right] \in \mathrm{PSL}(2,\mathbf{C}), \quad \operatorname{Re}(\lambda) > 0,$$

and $\operatorname{Re}(\lambda)$ is the length of the corresponding closed geodesic, and $\operatorname{Im}(\lambda)$ its geometric torsion. The parameter λ is the so-called *complex length*, and is only well defined up to $2\pi i$. We will regard this as a function

$$\begin{array}{rcl} \lambda\colon \mathcal{C}(M) & \to & \mathbf{C}/\langle 2\pi i\rangle \\ \varphi & \mapsto & \lambda(\varphi) = l(\varphi) + i\operatorname{torsion}(\varphi). \end{array}$$

In order to avoid the $2\pi i$ indeterminacy, we will work with the exponential of this map.

Definition. The *(prime) complex length spectrum* of M, denoted as $\mu_{\mathrm{sp}}\, M$, is the measure on $\mathbf{C}$ defined by

$$\mu_{\mathrm{sp}}\, M = \sum_{\varphi \in \mathcal{PC}(M)} \delta_{e^{\lambda(\varphi)}},$$

where δ_x is the Dirac measure centered at x. In other words, $\mu_{\mathrm{sp}}\, M$ is the image measure of the counting measure in $\mathcal{PC}(M)$ under the exponential of the complex length function. The *(prime) length spectrum* of M, denoted as $\mu_{\mathrm{lsp}}\, M$, is the measure on $\mathbf{R}$ defined by

$$\mu_{\mathrm{lsp}}\, M = \sum_{\varphi \in \mathcal{PC}(M)} \delta_{l(\varphi)}.$$

For instance, we have

$$\#\{\varphi \in \mathcal{PC}(M) \mid a < l(\varphi) < b\} = \mu_{\mathrm{sp}}\, M\{z \in \mathbf{C} \mid e^a < |z| < e^b\}.$$

Remark. The prime complex length spectrum is usually regarded as a collection of numbers and multiplicities. This is of course equivalent to the definition made above; however, we think that some of the results that we will present in what follows are better expressed in these terms.

The following properties of $\mu_{\mathrm{sp}}\, M$ are immediately implied by Lemma 4.3, and the fact that a closed geodesic cannot be contained in a cusp.

Proposition 4.4. *Assume that M has finite volume, then the following assertions hold:*

1. *The measure $\mu_{\mathrm{sp}}\, M$ is locally finite with discrete support. In particular, it is a Radon measure on the complex plane.*

2. *Let $N_1, \dots, N_j$ be cusps of M in such a way that $K = M \setminus \cup_{1 \leq j \leq n} N_j$ is compact. Then for all $R > 1$,*

$$\mu_{\mathrm{sp}}\, M\big(\{|z| \leq R\}\big) \leq C_M R^2,$$

   where $C_M = \pi \frac{e^{8 \operatorname{diam} K}}{\operatorname{Vol} K}$.

Next we want to analyse the complex length spectrum as a map

$$M \mapsto \mu_{\mathrm{sp}}\, M.$$

The domain of this map will be the set $\mathcal{M}$ of all (isometry classes of) oriented complete hyperbolic 3-manifolds of finite volume. This set is naturally endowed with the geometric topology, which is briefly discussed in next subsection.

On the other hand, the target of this map will be $M(\mathbf{C} \setminus \overline{D})$, the vector space of $\mathbf{C}$-valued Radon measures defined on the complementary of the disk $\overline{D} = \{z \in \mathbf{C} \mid |z| \leq 1\}$. We will endow $M(\mathbf{C} \setminus \overline{D})$ with the topology of the weak convergence. Thus a sequence $\{\mu_n\}$ converges

weakly to μ in $M(\mathbf{C}\setminus\overline{D})$ if for every continuous function f with compact support contained in $\mathbf{C}\setminus\overline{D}$,

$$\lim_{n\to\infty}\int_{|z|>1} f(z)d\mu_n(z) = \int_{|z|>1} f(z)d\mu(z).$$

The aim of the rest of the section is essentially to prove that this map is continuous.

Theorem 4.5. *The map $\mu_{\mathrm{sp}}\colon \mathcal{M}\to M(\mathbf{C}\setminus\overline{D})$ is continuous.*

Remark. If instead of the space $M(\mathbf{C}\setminus\overline{D})$ we consider $M(\mathbf{C})$, then the above theorem is no longer true. For instance, let $M\in\mathcal{M}$ be a one-cusped manifold, and $M_{p/q}$ be the manifold obtained by a hyperbolic (p,q)-Dehn filling. Then $\{M_{p/q}\}_{(p,q)}$ converges to M as p^2+q^2 goes to infinity. However, the sequence of the corresponding measures do not even converge. To see this, let $\pm\varphi_{p/q}$ be the two (oriented) core prime geodesics added in the Dehn filling. Then the length of $\varphi_{p/q}$ goes to zero, and the geometric torsion is dense in $\mathbf{R}/2\pi\mathbf{Z}$, what implies that this sequence of measures does not converge. Restricting our attention to $M(\mathbf{C}\setminus\overline{D})$ we avoid these phenomena. Nevertheless, this bad behaviour is the worst that can happen; this is expressed in the following result.

Theorem 4.6. *Let $M\in\mathcal{M}$ with $k>0$ cusps, and $\{M_n\}$ be a sequence converging to M in $\mathcal{M}$. Assume that the number of cusps of M_n is eventually constant and equal to l. Then the sequence of real length spectrum measures $\{\mu_{\mathrm{lsp}}\,M_n\}$ converges weakly in $M(\mathbf{R})$ to the measure*

$$\mu_{\mathrm{lsp}}\,M + 2(k-l)\delta_0.$$

Both Theorems 4.5 and 4.6 will be proved in Subsection 4.4 after having discussed the geometric topology.

4.3 The geometric topology

Most of the material in this subsection is based on [CEM06].

Let $\mathcal{MF}$ be the set of (isometry classes of) oriented complete hyperbolic 3-manifolds of finite volume and with a baseframe. Thus an element of $\mathcal{MF}$ is a pair (M,E), where E is an orthonormal frame based at some point p in the oriented hyperbolic 3-manifold M of finite volume.

Remark. Our notation differs from [CEM06], where $\mathcal{MF}$ is defined without the finite volume restriction.

If we fix a base frame on hyperbolic space $\mathbf{H}^3$, then the holonomy representation of a member of $\mathcal{MF}$ is unambiguously defined (i.e. not only up to conjugation). Therefore, $\mathcal{MF}$ is in one-to-one correspondence with the set of discrete torsion-free subgroups of $\mathrm{PSL}(2,\mathbf{C})$ with *finite co-volume*. The latter set is endowed with the *geometric topology*. We recall this definition in the general context of Lie groups, see [Thu].

Definition. A sequence $\{\Gamma_n\}$ of closed subgroups of a Lie group G *converges geometrically* to a group Γ if the following conditions are satisfied:

1. Each $\gamma \in \Gamma$ is the limit of a sequence $\{\gamma_n\}$, with $\gamma_n \in \Gamma_n$.
2. The limit of every convergent sequence $\{\gamma_{n_j}\}$, with $\gamma_{n_j} \in \Gamma_{n_j}$, is in Γ (n_j is an increasing sequence of natural numbers).

Two related spaces are $\mathcal{MB}$ and $\mathcal{M}$. The former is obtained by forgetting the frame, but retaining the basepoint, and the latter by forgetting both the frame and the basepoint. Both sets are endowed with the quotient topology given by the corresponding forgetful maps.

The following results are well known. They will play an important role in the following subsections. For a proof, see [CEM06]

Lemma 4.7. *Let* $\mathrm{inj}_R(M,p)$ *be the infimum of the injectivity radius on the ball* $B_R(p) \subset M$*. Then for any* $R > 0$ *the map* $\mathrm{inj}_R \colon \mathcal{MB} \to (0,\infty)$ *is continuous.*

Lemma 4.8. *Let* $\epsilon > 0$ *smaller than the Margulis constant. Let* $\{M_n\}$ *be a sequence converging to* M *in* $\mathcal{M}$*. Then there exists a uniform bound on the diameter of the thick parts* $\{M_{n,[\epsilon,\infty)}\}$*.*

Theorem 4.9 (Jorgensen)**.** *The map* $\mathrm{Vol}\colon \mathcal{M} \to \mathbf{R}$ *that assigns to each manifold its volume is continuous.*

The following theorem, due to Thurston, describes how is a non-trivial convergence sequence in $\mathcal{M}$. We recall that we are assuming that all manifolds have finite-volume.

Theorem 4.10 (Thurston)**.** *Let* $\{M_n\}$ *be a sequence converging to* M *in* $\mathcal{M}$*. Assume that* $\{M_n\}$ *is not eventually constant, and that* M *has* k *cusps. Then* M_n *is obtained by hyperbolic Dehn surgery* $M_{p_{1,n}/q_{1,n},\ldots,p_{k,n}/q_{k,n}}$*, with* $p_{i,n}^2 + q_{i,n}^2 \to \infty$*, as* $n \to \infty$*.*

Corollary 4.11. *Let* $\{(M_n, E_n)\}$ *be a sequence converging to* (M,E) *in* $\mathcal{MF}$*. Then, for* n *large enough, we have a commutative diagram,*

$$\begin{tikzcd} \pi_1(M,p) \arrow[r, "\rho_n"] \arrow[d, "i_*^n"] & \mathrm{PSL}(2;\mathbf{C}) \\ \pi_1(M_n,p_n) \arrow[ur, "\mathrm{Hol}_{M_n}"'] & \end{tikzcd}$$

Moreover, the sequence of representations $\{\rho_n\}$ *converges to* Hol_M *both algebraically (that is, for all* $\sigma \in \pi_1(M,p)$ *the sequence* $\{\rho_n(\sigma)\}$ *converges to* $\mathrm{Hol}_M(\sigma)$*), and geometrically (that is, the sequence of discrete groups* $\{\rho_n(\pi_1(M,p))\}$ *converges geometrically to* $\mathrm{Hol}_M(\pi_1(M,p))$*).*

It can be proved that if a sequence $\{(M_n,p_n)\}$ converges to (M,p) in $\mathcal{MB}$, then it also converges to (M,p) in the pointed Hausdorff-Gromov sense, see [CEM06].

Next we want to give the following *ad hoc* definition concerning the convergence of geodesics.

Definition. With the above notation, we will say that a sequence of parametrised closed geodesics $\{\varphi_n\colon [0,1] \to M_n\}$ converges to $\varphi\colon [0,1] \to M$ if for all n there is a lift $\tilde{\varphi}_n\colon [0,1] \to \mathbf{H}^3$ of φ_n (respect to the covering map π_n) such that the sequence of maps $\{\tilde{\varphi}_n\}$ converges pointwise to a lift of φ (respect to the covering map π).

Remark. The above definition coincides with the more general (and natural) definition of convergence of maps $\{f_n\colon X_n \to Y_n\}$, where $\{X_n\}$ and $\{Y_n\}$ are sequences of compact metric space converging in the Hausdorff-Gromov sense to X and Y respectively, see [GP91].

With the above definition, it is quite obvious that the limit of parametrised closed geodesics is also a geodesic whose length is the limit of the lengths of the converging geodesics.

Definition. We will say that a sequence $\{\varphi_n\}$ of closed geodesics, with $\varphi_n \in \mathcal{C}(M_n)$, converges to $\varphi \in \mathcal{C}(M)$ if for all n we can choose parametrisations of φ_n converging to a parametrisation of φ (in the sense of the above definition).

Again the following result holds in a more general context, see [GP91]. Its proof in our case is quite obvious.

Theorem 4.12 (Ascoli-Arzela, Grove-Petersen.)**.** *Let $R > 0$ and $\{\varphi_n\}$ be a sequence of closed geodesics with $\varphi_n \subset B_R(p_n) \subset (M_n, p_n)$. If there exists a common upper bound on the lengths of $\{\varphi_n\}$, then $\{\varphi_n\}$ has a converging subsequence.*

4.4 Proof of the continuity

In this subsection we want to prove the continuity of the complex length spectrum as a map from $\mathcal{M}$ to $M(\mathbf{C} \setminus \overline{D})$. A first obvious observation is that we can assume that this map is defined from $\mathcal{MF}$ to $M(\mathbf{C} \setminus \overline{D})$, since the topology of $\mathcal{M}$ is the quotient topology coming from the forgetful map $\mathcal{MF} \to \mathcal{M}$.

Hereafter $\{(M_n, E_n)\}$ will denote a sequence converging to $\{(M, E)\}$ in $\mathcal{MF}$. In order to simplify notation, we will write μ_n and μ_∞ for $\mu_{\mathrm{sp}}\, M_n$ and $\mu_{\mathrm{sp}}\, M$, respectively. We want to prove that the sequence of measures $\{\mu_n\}$ converges to $\mu_\infty M$ in $M(\mathbf{C} \setminus \overline{D})$. Our first task is to translate this into geometrical terms.

Recall from last subsection that we have a commutative diagram,

$$\begin{tikzcd} \pi_1(M,p) \arrow[r, "\rho_n"] \arrow[d, "i_*^n"] & \mathrm{PSL}(2;\mathbf{C}) \\ \pi_1(M_n,p_n) \arrow[ur, "\mathrm{Hol}_{M_n}"'] & \end{tikzcd}$$

Furthermore, the sequence of representations $\{\rho_n\}$ converges both algebraically and geometrically to Hol_M. Let $\sigma \in \pi_1(M,p)$ be a hyperbolic element. The algebraic convergence of $\{\rho_n\}$ implies that $\rho_n(\sigma)$ is also of hyperbolic type for large n (it follows for instance from the

fact that the set of hyperbolic isometries is open in $\mathrm{PSL}(2,\mathbf{C})$). As a consequence, for large n, the conjugacy class of $i_*^n(\sigma)$ defines a closed geodesic in M_n; moreover, the complex length of $\rho_n(\sigma)$ is close to that of $\mathrm{Hol}_M(\sigma)$.

Let $0 < a < b$. Then, for large n, the map $i_*^n\colon \pi_1(M,p) \to \pi_1(M_n,p_n)$ gives a well defined map

$$\iota_{a,b,n}\colon \{\varphi \in \mathcal{C}(M) \mid a < l(\varphi) < b\} \to \{\varphi \in \mathcal{C}(M_n) \mid a < l(\varphi) < b\}\,.$$

Lemma 4.13. *Assume that for all* $0 < a < b$ not in the real length spectrum *of* M *there exists* $N(a,b)$ *such that for all* $n > N(a,b)$ *the map* $\iota_{a,b,n}$ *is a bijection when restricted to prime geodesics. Then* $\{\mu_n\}$ *converges weakly to* μ_∞*.*

Proof. For two real numbers $a < b$ put $D_{a,b} = \{z \in \mathbf{C} \mid e^a < |z| < e^b\}$. Let f be a continuous function with compact support contained in the exterior of the unit disk. Take $1 < a < b$ such that $\operatorname{supp} f \subset D_{a,b}$, with both a and b not in the real length spectrum of M. Let $A = \{\varphi_1, \ldots, \varphi_k\}$ be the set of prime closed geodesics in M with complex spectrum in $D_{a,b}$. Therefore, we have

$$\int_{|z|>1} f(z)d\mu_\infty(z) = \int_{D_{a,b}} f(z)d\mu_\infty(z) = \sum_{i=1}^{k} f(\lambda(\varphi_i)).$$

By hypothesis, for $n > N(a,b)$, we have

$$\int_{|z|>1} f(z)d\mu_n(z) = \int_{D_{a,b}} f(z)d\mu_n(z) = \sum_{i=1}^{k} f(\lambda_n(\iota_{n,a,b}(\varphi_i))),$$

where λ_n is the complex length function of M_n. The algebraic convergence implies

$$\lim_{n\to\infty} \lambda_n(\iota_{n,a,b}(\varphi_i)) = \lambda(\varphi_i),$$

and the continuity of f give

$$\lim_{n\to\infty} \int_{|z|>1} f(z)d\mu_n(z) = \int_{|z|>1} f(z)d\mu_\infty(z).$$

Hence, μ_n converges weakly to μ_∞. $\square$

Next we want to prove that the hypothesis of the above lemma are satisfied. Hereafter a and b will denote two fixed positive real numbers not in the length spectrum of M with $a < b$. We will write ι_n instead of $\iota_{a,b,n}$.

The following lemma is an immediate consequence of the convergence of $\{(M_n,E_n)\}$ to (M,E).

Lemma 4.14. *Let* $\varphi \in \mathcal{C}(M)$*. Then the sequence of closed geodesics* $\{\iota_n(\varphi)\}$ *converges to* φ*.*

Proposition 4.15. *Let $\varphi_1, \varphi_2 \in \mathcal{C}(M)$. If $\varphi_1 \neq \varphi_2$ then, for n large enough, $\iota_n(\varphi_1) \neq \iota_n(\varphi_2)$.*

Proof. We have $d(\varphi_1, \varphi_2) > 0$. The above lemma then implies that for large n also

$$d(\iota_n(\varphi_1), \iota_n(\varphi_2)) > 0.$$

□

Proposition 4.16. *If $\varphi \in \mathcal{C}(M)$ is prime, then, for n large enough, $\iota_n(\varphi)$ is also prime.*

Proof. Take $R > 0$ such that $\iota_n(\varphi) \subset B_R(p_n)$ for all n. If the lemma were false, then (up to a subsequence) for all n, $\iota_n(\varphi) = k_n\psi_n$ for some integer $k_n \geq 2$ and some $\psi_n \in \mathcal{PC}(M_n)$. By Lemma 4.7, the injectivity radius on $B_R(p_n)$ is uniformly bounded from below away from zero; hence, k_n must be bounded from above. Therefore, (up to a subsequence) for all n, $\iota_n(\varphi) = k\psi_n$, for some fixed $k \geq 2$. The geodesics $\{\psi_n\}$ have bounded length and are contained in $B_R(p_n)$; hence, by Ascoli-Arzela (up to a subsequence) they converge to a geodesic ψ which satisfies $\varphi = k\psi$, contradicting the primality of φ. □

These two preceding results imply that, for large n, ι_n gives an injective map

$$\{\varphi \in \mathcal{PC}(M) \mid a < l(\varphi) < b\} \to \{\varphi \in \mathcal{PC}(M_n) \mid a < l(\varphi) < b\},$$

Next we want to prove that, for a larger n, this map is surjective. We will proceed by contradiction using an Arzela-Ascoli argument. Before doing that, we need to prove that we have a control on prime closed geodesics in M_n whose lengths are in (a, b). This is the content of the following result, which is just an application of the thick-thin decomposition of a complete finite-volume hyperbolic manifold.

Lemma 4.17. *There exists $R > 0$ such that, for all n, any* prime *closed geodesic in (M_n, p_n) of length in (a, b) is contained in $B_R(p_n)$.*

Proof. Let $\epsilon > 0$ be smaller than $a/2$ and the Margulis constant. If necessary, take a smaller $\epsilon > 0$ to guarantee that $p_n \in M_{n,[\epsilon,\infty)}$. If φ is a closed geodesic in M_n of length $a < l(\varphi)$, then φ intersects the ϵ-thick part $M_{n,[\epsilon,\infty)}$ (otherwise φ would be the core of a Margulis tube in $M_{n,(0,\epsilon)}$, and the injectivity radius in that tube would be achieved by the curve φ, so $a/2 < l(\varphi_n)/2 \leq \epsilon$, what is absurd). The result then follows from the fact that the diameter of $M_{n,[\epsilon,\infty)}$ is uniformly bounded on n. □

Lemma 4.18. *There exists N such that for all $n > N$ the following holds: if φ_n is a prime closed geodesic in M_n of length $a < l(\varphi_n) < b$, then there exists a prime closed geodesic on M of length $a < l(\varphi) < b$ with $\varphi_n = \iota_n(\varphi)$.*

Proof. Assume that the lemma is false. Up to a subsequence, for all n there exists a prime closed geodesic φ_n on M_n with $l(\varphi_n) \in (a,b)$ such that $\varphi_n \neq \iota_n(\psi)$, for all $\psi \in \mathcal{PC}(M)$ of length in (a,b). Take the R given by Lemma 4.17. By the continuity of the injectivity radius, there exists a uniform lower bound $\epsilon > 0$ for the injectivity radius on $B_R(p_n)$. Therefore, by Proposition 4.2, for all $\psi \in \mathcal{PC}(M)$ of length in (a,b),

$$d(\iota_n(\psi), \varphi_n) > \epsilon.$$

Up to a subsequence, $\{\varphi_n\}$ converges to a closed geodesic φ in M. It is easily seen that φ must be prime. Since $l(\varphi) \in (a,b)$ (recall that a and b do not belong to the length spectrum of M), the above inequality says that

$$d(\iota_n(\varphi), \varphi_n) > \epsilon.$$

It contradicts the fact that both $\{\varphi_n\}$ and $\{\iota_n(\varphi)\}$ converge to φ. □

Proof of Theorem 4.5. Propositions 4.15 and 4.16 prove that ι_n is injective, and Lemma 4.18 states that ι_n is surjective. Then Lemma 4.13 proves that $\{\mu_n\}$ converges to μ_∞ weakly. □

It remains to prove Theorem 4.6. In order to do it, we can assume that M has k cusps, and that the sequence $\{M_n\}$ converging to M in $\mathcal{M}$ is obtained by performing Dehn fillings on l $(\leq k)$ fixed cusps of M. We must prove that the sequence of (real) length spectrum measures $\{\mu_{\text{lsp}}\, M_n\}$ converges in $M(\mathbf{R})$ to

$$\mu_{\text{lsp}}\, M + 2(k-l)\delta_0.$$

By Theorem 4.5, it is enough to prove that there exists $\delta > 0$ smaller than the length of the shortest geodesic on M such that

$$\lim_{n\to\infty} \mu_{\text{lsp}}\, M_n([0,\delta)) = 2(k-l).$$

In geometrical terms, it is equivalent to the following well known result.

Lemma 4.19. *Let $\{\pm\varphi_n^1, \ldots, \pm\varphi_n^l\}$ be the core geodesics (oriented and prime) in M_n added on the Dehn filling. Let δ_s be the length of the shortest geodesic in M, and $\delta \in (0, \delta_s)$. Then for large n the only prime closed geodesics in M_n of length $< \delta$ are the core geodesics.*

Proof. Take $\epsilon > 0$ smaller than both the Margulis constant and $\delta/2$. Then $M_{(0,\epsilon)}$ consists only of cusps. Since $l(\varphi_n^i)$ goes to zero as n goes to infinity, for large n, all the geodesics φ_n^i are in $M_{n,(0,\epsilon)}$. Let T_n^i be the Margulis tube corresponding to φ_n^i, and $\{C_n^{l+1}, \ldots, C_n^k\}$ be the cusps components of $M_{n,(0,\epsilon)}$ corresponding to the non-deformed cusps. Let

$$F_n = T_n^1 \cup \cdots \cup T_n^l \cup C_n^{l+1} \cup \cdots \cup C_n^k \subset M_{n,(0,\epsilon)}.$$

For large n, $M_{n,[\epsilon,\infty)}$ is homeomorphic to $M_{[\epsilon,\infty)}$; in particular, $M_{n,(0,\epsilon]}$ has k boundary components. It implies that, for large n, $F_n = M_{n,(0,\epsilon)}$, and the result follows. □

In the following section, we will need the following improvement of Theorem 4.5.

Proposition 4.20. *Let* $f\colon \mathbf{C} \to \mathbf{C}$ *be a continuous function with* $\operatorname{supp} f$ *not necessarily compact but contained in* $\mathbf{C} \setminus \overline{D}$. *Assume that there exists* $\epsilon > 0$, *and* $K > 0$ *such that*

$$|f(z)| \leq \frac{K}{|z|^{2+\epsilon}},$$

for all $z \in \mathbf{C}$. *Then we have:*

1. *For any* $M \in \mathcal{M}$,
$$\int_{|z|>1} |f(z)| d\,\mu_{\mathrm{sp}}\, M(z) < \infty.$$

2. *If* $\{M_n\}$ *converges to* M *in* $\mathcal{M}$, *then*
$$\lim_{n\to\infty} \int_{|z|>1} f(z) d\,\mu_{\mathrm{sp}}\, M_n(z) = \int_{|z|>1} f(z) d\,\mu_{\mathrm{sp}}\, M(z).$$

Proof. Take δ be the Margulis constant. Then for all $M \in \mathcal{M}$ any prime closed geodesic on M of length $\geq 2\delta$ intersects the thick part $M_{[\delta,\infty)}$. Let $M \in \mathcal{M}$, and put $\mu = \mu_{\mathrm{sp}}\, M$. Fix $R \gg 1$. By Lemma 4.3, we have

$$\mu(\{e^{2\delta} \leq |z| \leq R\}) \leq CR^2,$$

where $C = \pi \frac{e^{8 \operatorname{diam} M_{[\delta,\infty)}}}{\operatorname{Vol} M_{[\delta,\infty)}}$. Then we have,

$$\begin{aligned}
\int_{|z|\geq R} |f(z)| d\mu(z) &= \sum_{k=0}^{\infty} \int_{R2^k \leq |z| < R2^{k+1}} |f(z)| d\mu(z) \\
&\leq \sum_{k=0}^{\infty} \int_{R2^k \leq |z| < R2^{k+1}} \frac{K}{|z|^{2+\epsilon}} d\mu(z) \\
&\leq \sum_{k=0}^{\infty} \frac{K}{(R2^k)^{2+\epsilon}} \int_{R2^k \leq |z| < R2^{k+1}} d\mu(z) \\
&\leq \sum_{k=0}^{\infty} \frac{K}{(R2^k)^{2+\epsilon}} C(R2^{k+1})^2 \\
&= \frac{KC}{R^\epsilon} \sum_{k=0}^{\infty} \frac{2^{2k+2}}{2^{k(2+\epsilon)}} = \frac{4KC}{R^\epsilon} \frac{1}{1-\frac{1}{2^\epsilon}} = \frac{C'}{R^\epsilon},
\end{aligned}$$

where C' is a constant depending only on C, K, and ϵ. The first assertion is then proved. Now let $\{M_n\}$ converging to M in $\mathcal{M}$. Let us put $\mu_n = \mu_{\mathrm{sp}}\, M_n$. Since both $\operatorname{diam} M_{n,[\delta,\infty)}$ and

Vol M_n are uniformly bounded on n, the first assertion implies that there exist a constant C'' such that for all n

$$\int_{|z|\geq R} |f(z)| d\mu_n(z) \leq \frac{C''}{R^\epsilon}.$$

Thus we have

$$\begin{aligned}
\left| \int_{|z|>1} f(z)(d\mu_n(z) - d\mu(z)) \right| &\leq \left| \int_{1<|z|<R} f(z)(d\mu_n(z) - d\mu(z)) \right| \\
&+ \left| \int_{|z|\geq R} f(z)(d\mu_n(z) - d\mu(z)) \right| \\
&\leq \left| \int_{1<|z|<R} f(z)(d\mu_n(z) - d\mu(z)) \right| + \frac{C'' + C'}{R^\epsilon}.
\end{aligned}$$

Theorem 4.5 shows that

$$\lim_{n\to\infty} \left| \int_{|z|>1} f(z)(d\mu_n(z) - d\mu(z)) \right| \leq \frac{C'' + C'}{R^\epsilon}.$$

Since R is arbitrary and independent of both C and C'', the left hand side of the above equation must vanish. $\square$

4.5 Spin complex length spectrum

Let M be an oriented complete hyperbolic 3-manifold. Let η be a spin structure on M, and consider the associated lift of the holonomy representation,

$$\mathrm{Hol}_{(M,\eta)} \colon \pi_1(M,p) \to \mathrm{SL}(2,\mathbf{C}).$$

If $\gamma \in \pi_1(M,p)$ is of hyperbolic type then,

$$\mathrm{Hol}_{(M,\eta)}(\gamma) \sim \begin{pmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbf{C}), \quad \mathrm{Re}(\lambda) > 0.$$

The *spin complex length* of γ is by definition the parameter $\lambda \in \mathbf{C}/\langle 4\pi i\rangle$. Hence, in contrast to the usual complex length, $e^{\lambda/2}$ is well defined (we have a well defined sign given by the lift of the holonomy). We propose the following definition.

Definition. The (prime) spin complex length spectrum of (M,η) is defined by

$$\mu_{\mathrm{sp}}(M,\eta) = \sum_{\varphi\in\mathcal{PC}(M)} \delta_{e^{\lambda(\varphi)/2}},$$

where δ_x is the Dirac measure centered at x.

Remark. The image measure of $\mu_{\mathrm{sp}}(M,\eta)$ under the function $z \mapsto z^2$ is $\mu_{\mathrm{sp}}\, M$.

The results obtained for the length spectrum in the previous subsections extend in a natural way for the spin complex length spectrum. In order to do that we must consider the space $\mathcal{MSF}$ of spin hyperbolic manifold with a baseframe. In this case we have the identification between $\mathcal{MSF}$ and the space of discrete torsion-free subgroups of $\mathrm{SL}(2,\mathbf{C})$ with finite co-volume. We topologize $\mathcal{MSF}$ in such a way that this identification becomes an homeomorphism. The quotient spaces $\mathcal{MSB}$ and $\mathcal{MS}$ are then defined as in the non-spin case.

Theorem 4.21. *The map $\mu_{\mathrm{sp}}\colon \mathcal{MS} \to M(\mathbf{C}\setminus \overline{D})$ is continuous.*

As in the non-spin case, we can improve the continuity in the following sense. Notice that the condition on the decay at infinity must be replaced, since the measure of the ball $B_R(0) \subset \mathbf{C}$ under the measure $\mu_{\mathrm{sp}}(M,\eta)$ is equal to the measure of the ball $B_{R^2}(0)$ under the measure $\mu_{\mathrm{sp}}\, M$.

Proposition 4.22. *Let $f\colon \mathbf{C}\to\mathbf{C}$ be a continuous function with support contained in $|z|>1$. Assume that there exists $\epsilon>0$, and $K>0$ such that*

$$|f(z)| \leq \frac{K}{|z|^{4+\epsilon}},$$

for all $|z|>1$. If $\{(M_n,\eta_n)\}$ converges to (M,η) in $\mathcal{MS}$, then

$$\int_{|z|>1} |f(z)| d\mu(z), \int_{|z|>1} |f(z)| d\mu_n(z) < \infty,$$

and

$$\lim_{n\to\infty} \int_{|z|>1} f(z) d\mu_n(z) = \int_{|z|>1} f(z) d\mu(z).$$

Where $\mu = \mu_{\mathrm{sp}}(M,\eta)$ and $\mu_n = \mu_{\mathrm{sp}}(M_n,\eta_n)$

5 Asymptotic behavior

The aim of this section is to establish the asymptotic behavior of the n-th dimensional hyperbolic Reidemeister torsion. More concretely, we will prove the following result.

Theorem 5.1. *Let M be a connected complete hyperbolic 3-manifold of finite volume. Then*

$$\lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k+1}(M)|}{(2k+1)^2} = -\frac{\mathrm{Vol}(M)}{4\pi}.$$

In addition, if η is an acyclic spin structure on M, then

$$\lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k}(M,\eta)|}{(2k)^2} = -\frac{\mathrm{Vol}(M)}{4\pi}.$$

For a compact manifold, the above result is due to Müller, see [Mül]. In this case, we can consider $\tau_n(M;\eta)$ for all n (i.e. there is no need to consider the normalized torsion $\mathcal{T}_n(M,\eta)$).

Theorem 5.2 (Müller, [Mül])**.** *Let M be a connected oriented hyperbolic 3-manifold. Then, for every spin structure η on M,*

$$\lim_{n\to\infty} \frac{\log|\tau_n(M;\eta)|}{n^2} = -\frac{\mathrm{Vol}(M)}{4\pi}.$$

The proof given by Müller is based on the fact that the Reidemeister torsion coincides with the Ray-Singer analytic torsion for a compact manifold. Since the Ray-Singer torsion is not even defined for non-compact manifolds, it seems difficult to adapt Müller's proof to the non-compact case. Nevertheless, Müller's techniques are still powerful in the non-compact case, and will play a crucial role in our proof of Theorem 5.1. Roughly speaking, our approach will consist in approximating the cusp manifold M by compact manifolds obtained by hyperbolic Dehn filling; then we will apply Müller's theorem to these compact manifolds and the surgery formula for the torsion stated in Section 3. The continuity of the (spin) complex length spectrum established in Section 4 will allow us to handle this limit process.

The distribution of this section is as follows. The first subsection is an exposition of the notions concerning the Ray-Singer analytic torsion and Ruelle zeta functions that will be needed in the subsequent subsections; that subsection ends with Wotzke's theorem in dimension three, which gives the relationship between Ruelle zeta functions and the Reidemeister torsion invariants that we are studying. In the second subsection, we will state the theorem by Müller from which he deduces the asymptotic behaviour for the compact case. That theorem establishes a formula for the Ray-Singer analytic torsion, which will be the essential ingredient for the proof of Theorem 5.1 given in the last subsection.

5.1 Ray-Singer analytic torsion and Ruelle zeta functions

Let M be a differentiable closed n-manifold with a Riemannian metric g. Let us assume that we have an *acyclic* orthogonal (or unitary) representation of the fundamental group

$$\rho\colon \pi_1 M \to \mathrm{O}(n).$$

The analytic Ray-Singer torsion $T(M;\rho)$, introduced by Ray and Singer in the seminal paper [RS71], is a certain weighted alternating product of regularized determinants of the Laplacians

$$\Delta^q\colon \Omega^q(M;E_\rho) \to \Omega^q(M;E_\rho).$$

A theorem proved in [RS71] states that the Ray-Singer torsion is independent of the metric chosen. Hence, it is usually denoted simply as $T(M;\rho)$, without making reference to the metric g.

In the paper mentioned above, Ray and Singer conjectured that the Reidemeister torsion $\tau(M;\rho)$ agrees with the analytic torsion $T(M;\rho)$. This conjecture was proved independently

by Cheeger and Müller in [Che79] and [Mül78] respectively. In [Mül93], Müller extended the definition of the analytic torsion to unimodular representations

$$\rho\colon \pi_1 M \to \mathrm{SL}(n, \mathbf{C}).$$

As in the orthogonal case, this definition requires a Riemannian metric, but, in contrast to the orthogonal case, this new analytic torsion is only metric independent for odd dimensions. In that paper, Müller also proved that both the analytic torsion and the Reidemeister torsion agree for an odd dimensional closed manifold.

An important part of this story concerns the relation between the Ray-Singer torsion and Ruelle zeta functions for a compact negatively curved manifold M. Since it will play a crucial role in the proof of our main theorem, we will spend the rest of this subsection to explain it.

Let Γ be a torsion free co-compact subgroup of $\mathrm{Isom}^+ \mathbf{H}^n$, and let $M = \mathbf{H}^n/\Gamma$ be the corresponding hyperbolic manifold. The classical Ruelle zeta function associated to M is formally defined as

$$R(s) = \prod_{[\gamma]\in \mathrm{PC}(\Gamma)} \left(1 - e^{-sl(\gamma)}\right),$$

where $l(\gamma)$ is the length of the prime oriented closed geodesic defined by the prime conjugacy class $[\gamma]$ of Γ. The region of convergence of $R(s)$ can be determined using the asymptotic behaviour of the number of closed geodesics of length less or equal than a given value. To that end, define $P(t)$ as

$$P(t) = \#\Big\{[\gamma] \in \mathrm{P}\Gamma \mid l(\gamma) \leq t\Big\}.$$

Margulis studied the function $P(t)$ for a closed manifold of negative curvature in [Mar69]. Among other things, he proved that

$$\lim_{t\to\infty} \frac{P(t)}{e^{ht}/ht} = 1,$$

where h is the topological entropy of the geodesic flow. The topological entropy of a hyperbolic manifold of dimension n is $h = n - 1$. Using Marguli's result, the region of convergence of $R(s)$ is easily seen to be

$$\Big\{s \in \mathbf{C} \mid \mathrm{Re}(s) > n - 1\Big\}.$$

In [Fri86], Fried gave the following generalization on the definition of the Ruelle zeta function. Given an orthogonal representation $\rho\colon \pi_1(M) \to O(d)$, which need not to be acyclic, the twisted Ruelle zeta function associated to ρ is defined as

$$R_\rho(s) = \prod_{[\gamma]\in \mathrm{P}\Gamma} \det\left(\mathrm{Id} - \rho(\gamma)e^{-sl(\gamma)}\right).$$

The region of convergence of $R_\rho(s)$ is the same as the one of the classical Ruelle zeta function. In that same paper, Fried proved that $R_\rho(s)$ has a meromorphic extension to the whole

complex plane; moreover, if ρ is acyclic, then $R_\rho(s)$ is regular at $s=0$ and $|R_\rho(0)| = T(M;\rho)^2$ (if ρ is not acyclic, $R_\rho(s)$ can have a pole at $s=0$, and $T(M;\rho)^2$ is equal to the leading term of the Laurent expansion of $R_\rho(s)$ at the origin).

In a posterior paper [Fri95], Fried proved that for a general representation $\rho\colon \pi_1 M \to \mathrm{GL}(d;\mathbf{C})$ the twisted Ruelle zeta function $R_\rho(s)$ has also a meromorphic extension to the whole plane. However, he was not able to prove its relationship with the Ray-Singer analytic torsion. Nevertheless, three years later U. Bröcker proved in his thesis a similar result for representations of the fundamental group that are restrictions of finite-dimensional irreducible representations of $\mathrm{Isom}^+\mathbf{H}^n \cong \mathrm{SO}_0(n,1)$, see [Brö98]. According to Müller [Mül], the methods used by Bröcker are based on elaborate computations which are difficult to verify. Nonetheless, this problem has been overcome by Wotzke in his thesis [Wot08]. The following subsection is dedicated to state Wotzke's Theorem in dimension 3.

5.1.1 Wotzke's Theorem

Let (M,η) be a connected closed spin hyperbolic 3-manifold. If Γ is the image of $\pi_1(M,p)$ under the $\mathrm{Hol}_{(M,\eta)}$, then

$$M = \Gamma\backslash\,\mathrm{SL}(2;\mathbf{C})/\mathrm{SU}(2).$$

Let ρ be a *real* finite-dimensional representation of $\mathrm{SL}(2;\mathbf{C})$, regarded as a real Lie group. Denote by θ the Cartan involution of $\mathrm{SL}(2;\mathbf{C})$ with respect to $\mathrm{SU}(2)$, and put $\rho_\theta = \rho\circ\theta$. Let $E_\rho \to M$ be the flat vector bundle associated to ρ. Introduce some metric on E_ρ, and consider the Laplacians $\Delta^q\colon \Omega^r(M;E_\rho)\to\Omega^r(M;E_\rho)$.

Theorem 5.3 (Wotzke,[Wot08])**.** *With the above notation, the following assertions hold:*

1. *If ρ_θ is not isomorphic to ρ, then $R_\rho(s)$ is regular at $s=0$ and*
$$|R_\rho(0)| = T(M;\rho)^2.$$
2. *Assume that $\rho\circ\theta$ is isomorphic to ρ. If ρ is not trivial, then the order h_ρ at $s=0$ of $R_\rho(s)$ is given by*
$$h_\rho = 2\sum_{q=1}^{3}(-1)^q \dim\ker\Delta^q,$$
and for the trivial representation we have $h_\rho = 4 - 2\dim \mathrm{H}^1(M;\mathbf{R})$. The leading term of the Laurent expansion of $R_\rho(s)$ at $s=0$ is given by
$$T(M;\rho)^2 s^{h_\rho}.$$

Remark. The Cartan involution of the real Lie algebra $\mathfrak{sl}(2;\mathbf{C})$ is given by $\theta(X) = -\overline{X}^t$. It can be checked that a *complex* representation ρ of $\mathrm{SL}(2;\mathbf{C})$ is not equivalent to $\rho\circ\theta$.

5.2 Müller's Theorem

Let us retain the same notation as in the previous subsection (in particular, M will be assumed to be closed). For $n > 0$, let ρ_n be the composition of $\mathrm{Hol}_{(M,\eta)}$ with the $(n-1)$-th symmetric power of the standard representation of $\mathrm{SL}(2;\mathbf{C})$. Thus we have

$$\rho_n\colon \pi_1(M,p) \cong \Gamma \to \mathrm{SL}(n;\mathbf{C}).$$

Müller's theorem on the equivalence of the Reidemeister torsion and the Ray-Singer analytic torsion, implies that

$$T(M;\rho_{n,\eta}) = |\,\tau(M;\rho_n)|.$$

Let us denote by $R_n(s)$ the Ruelle zeta function associated to the representation ρ_n. Wotzke's Theorem then implies that

$$|R_{\rho_n}(0)| = |\,\tau(M;\rho_n)|^2.$$

Following [Mül], the Ruelle zeta function $R_{\rho_n}(s)$ can be expressed in terms of the following related Ruelle zeta functions,

$$R_k(s) = \prod_{[\gamma]\in \mathrm{PC}(\Gamma)} \big(1 - \sigma_k(\gamma) e^{-sl(\gamma)}\big),$$

where $\sigma_k(\gamma)$ is defined by

$$\sigma_k(\gamma) = e^{ki\,\mathrm{Im}\,\lambda(\gamma)/2} = e^{ki\theta(\gamma)/2},$$

with $\theta(\gamma)$ the geometric spin torsion of the closed geodesic defined by γ. A straightforward computation then shows that

$$R_{\rho_n}(s) = \prod_{k=0}^{n} R_{n-2k}(s-(n/2-k)).$$

The following theorem by Müller relates the Reidemeister torsion, Ruelle zeta functions and the volume of the manifold M.

Remark. Müller uses in [Mül] a different sign convention in the definition of the torsion, and he uses the notation τ_n to designate the representation coming from the n-th symmetric power, so his τ_n is our ρ_{n+1}.

Theorem 5.4 (**Müller [Mül]**)**.** *Let (M,η) be a closed spin hyperbolic 3-manifold. For $m \geq 3$, let ρ_m be the representation of $\pi_1 M$ defined above. Then we have the following equations,*

$$\begin{aligned}
\log\left|\frac{\tau(M;\rho_{2m+1})}{\tau(M;\rho_5)}\right| &= \sum_{k=3}^{m} \log|R_{2k}(k)| - \frac{1}{\pi}\,\mathrm{Vol}\,M\,(m(m+1)-6)\,,\\
\log\left|\frac{\tau(M,\eta;\rho_{2m})}{\tau(M,\eta;\rho_4)}\right| &= \sum_{k=2}^{m-1} \log\left|R_{2k+1}\left(k+\frac{1}{2}\right)\right| - \frac{1}{\pi}\,\mathrm{Vol}\,M(m^2-4)
\end{aligned}$$

Müller then deduces Theorem 5.2 from the following lemma, [Mül].

Lemma 5.5. *For a closed hyperbolic 3-dimensional manifold M there exists a constant $C > 0$, depending only on the manifold M, such that for all $m \geq 3$, we have*

$$\sum_{k=3}^{m} |\log|R_{2k}(k)|| < C, \quad \sum_{k=2}^{m-1} \left|\log|R_{2k+1}\left(k+\frac{1}{2}\right)\right| < C.$$

5.3 The noncompact case

Let (M, η) be a compactly approximable spin hyperbolic 3-manifolds of finite volume. In this subsection, we want to prove that Theorem 5.1 holds for (M, η) as well. We will do this by proving that Theorem 5.4 holds also for (M, η).

The definition of the Ruelle zeta function R_{ρ_n} for (M, η) is obvious if we define it in terms of prime closed geodesics; more concretely, we define

$$R_{\rho_n}(s) = \prod_{\varphi \in \mathcal{PC}(M)} \det\left(\mathrm{Id} - \rho_n(\varphi) e^{-sl(\varphi)}\right).$$

Of course, it makes sense also to define

$$R_k(s) = \prod_{\varphi \in \mathcal{PC}(M)} \left(1 - \sigma_k(\varphi) e^{-sl(\gamma)}\right).$$

The function $R_{\rho_n}(s)$ is related to the functions $R(s, \sigma_k)$ as in the compact case. The estimations concerning the growth of closed geodesics in M imply that $R(s, \sigma_k)$ converges for $\mathrm{Re}(s) > 2$. More accurate estimations will probably allow to conclude that the region of convergence of $R(s, \sigma_k)$ is exactly that half-plane. Therefore, the region of convergence of $R_{\rho_n}(s)$ contains the half-plane $\mathrm{Re}(s) > 2 + n/2$.

It is worth noting that the following equation holds,

$$\log\left|R_k\left(\frac{k}{2}\right)\right| = \int_{|z|>1} \log|1 - z^{-k}| d\,\mu_{\mathrm{sp}}(M, \eta)(z), \quad k \geq 3. \tag{5}$$

With the same notation as in Section 3, we have the following formula.

Lemma 5.6. *Let $(p, q) \in \mathcal{A}_{(M,\eta)}$, and $A = \{\pm\varphi_{p_1/q_1}, \ldots, \pm\varphi_{p_l/q_l}\}$ be the prime oriented core geodesics in $M_{p/q}$ added in the Dehn filling. For an integer $m \geq 3$, we have*

$$\log\left|\frac{\tau(M; \rho_{2m}^{p/q})}{\tau(M; \rho_4^{p/q})}\right| = -\frac{(m-2)(m+2)}{2} \sum_{i=1}^{l} l(\varphi_{p/q}^i) - \frac{1}{\pi} \mathrm{Vol}(M_{p/q})(m^2 - 4) + \sum_{k=2}^{m-1} B_{2k+1}^{p/q},$$

where

$$B_j^{p/q} = \sum_{\varphi \in \mathcal{PC}(M_{p/q}) \setminus A} \log\left|1 - e^{-j\lambda_{p/q}(\varphi)/2}\right|.$$

Proof. For the sake of simplicity we will prove it only for one-cusped manifolds. The surgery formula given by Lemma 3.7, yields

$$\log\left|\frac{\tau(M_{p/q};\rho_{2m}^{p/q})}{\tau(M;\rho_{2m}^{p/q})}\right| = \sum_{k=0}^{m-1}\log\left|\left(e^{\left(\frac{1}{2}+k\right)\lambda(\varphi_{p/q})}-1\right)\left(e^{-\left(\frac{1}{2}+k\right)\lambda(\varphi_{p/q})}-1\right)\right|.$$

It follows that,

$$\log\left|\frac{\tau(M_{p/q};\rho_{2m}^{p/q})\,\tau(M;\rho_{4}^{p/q})}{\tau(M_{p/q};\rho_{4}^{p/q})\,\tau(M;\rho_{2m}^{p/q})}\right| = \sum_{k=2}^{m-1}\log\left|\left(e^{\left(\frac{1}{2}+k\right)\lambda(\varphi_{p/q})}-1\right)\left(e^{-\left(\frac{1}{2}+k\right)\lambda(\varphi_{p/q})}-1\right)\right|.$$

Since $M_{p/q}$ is compact, Müller's Theorem 5.8 gives

$$\log\left|\frac{\tau(M_{p/q};\rho_{2m}^{p/q})}{\tau(M_{p/q};\rho_{4}^{p/q})}\right| = \sum_{k=2}^{m-1}\log|R_{2k+1}^{p/q}(k+\frac{1}{2})| - \frac{1}{\pi}\mathrm{Vol}(M_{p/q})(m^2-4).$$

From these last two equations, we get

$$\begin{aligned} -\log\left|\frac{\tau(M;\rho_{4}^{p/q})}{\tau(M;\rho_{2m}^{p/q})}\right| &= \sum_{k=2}^{m-1}\log|R_{2k+1}^{p/q}(k+\frac{1}{2})| - \frac{1}{\pi}\mathrm{Vol}(M_{p/q})(m^2-4) \\ &- \sum_{k=2}^{m-1}\log|e^{(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1||e^{-(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1|. \end{aligned}$$

Using the expression

$$\log|R_{2k+1}^{p/q}(k+\frac{1}{2})| = \log|1-e^{-(k+\frac{1}{2})\overline{\lambda(\varphi_{p/q})}}|^2 + B_{2k+1}^{p/q},$$

the above equation is written as

$$\begin{aligned} \log\left|\frac{\tau(M;\rho_{2m}^{p/q})}{\tau(M;\rho_{4}^{p/q})}\right| &= \sum_{k=2}^{m-1}\log\frac{|1-e^{-(k+\frac{1}{2})\overline{\lambda(\varphi_{p/q})}}|^2}{|e^{(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1||e^{-(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1|} \\ &- \frac{1}{\pi}\mathrm{Vol}(M_{p/q})(m^2-4) + \sum_{k=2}^{m-1}B_{2k+1}^{p/q}. \end{aligned}$$

We have,

$$\frac{|1-e^{-(k+\frac{1}{2})\overline{\lambda(\varphi_{p/q})}}|^2}{|e^{(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1||e^{-(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1|} = e^{-(\frac{1}{2}+k)\,\mathrm{Re}\,\lambda(\varphi_{p/q})}.$$

Hence,

$$\sum_{k=2}^{m-1} \log\left(\frac{|1-e^{-(k+\frac{1}{2})\overline{\lambda(\varphi_{p/q})}}|^2}{|e^{(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1||e^{-(\frac{1}{2}+k)\lambda(\varphi_{p/q})}-1|}\right) = -\frac{(m-2)(m+3)}{4} l(\varphi_{p/q}),$$

and the lemma follows. ☐

Lemma 5.7. *With the same notation as in the preceding lemma, for $k \geq 2$ we have*

$$\lim_{(p,q)\to\infty} B_k^{p/q} = \log\left|R_k\left(\frac{k}{2}\right)\right|.$$

Moreover, the following series is absolutely convergent

$$\sum_{k=5}^{\infty} \log\left|R_k\left(\frac{k}{2}\right)\right|.$$

Proof. Let δ be the length of the shortest closed geodesic in M. By Lemma 4.19, for (p,q) large enough, the only prime closed geodesics on $M_{p/q}$ of length less than $\delta/2$ are the core geodesics $A = \{\pm\varphi_{p_1/q_1}, \ldots, \pm\varphi_{p_l/q_l}\}$. In that case,

$$B_k^{p/q} = \sum_{\varphi\in\mathcal{PC}(M_{p/q})\setminus A} \log\left|1-e^{-k\lambda(\varphi)/2}\right| = \int_{|z|>e^{\delta/4}} \log|1-z^{-k}| d\mu_{p/q}(z).$$

where $\mu_{p/q} = \mu_{\mathrm{sp}}(M_{p/q}, \eta_{p/q})$. Now we want to apply Proposition 4.22. We shall show that for large $|z|$ we have

$$|\log|1-z^{-k}|| \leq \frac{C}{z^5}, \quad \text{ for } k \geq 5, \tag{6}$$

where C is some constant. First notice that for $w \in \mathbf{C}$ with $|w| < 1$ the following inequality holds

$$|\log|1-w|| \leq -\log|1-|w||.$$

On the other hand, for $|w|$ small enough,

$$-\log|1-|w|| \sim |w|.$$

Inequality 6 then follows easily from the last two inequalities. Therefore, we can use Proposition 4.22 to conclude that

$$\lim_{(p,q)\to\infty} B_k^{p/q} = \log\left|R_k\left(\frac{k}{2}\right)\right|.$$

Finally, if $\mu = \mu_{\rm sp}(M,\eta)$, we have

$$\begin{aligned} \sum_{k=5}^{\infty} \left| \log \left| R_k \left(\frac{k}{2} \right) \right| \right| &\leq \sum_{k=5}^{\infty} \int_{|z|>e^{\delta/2}} |\log \left|1-|z|^{-k}\right| | d\mu(z) \\ &\leq \sum_{k=5}^{\infty} \int_{|z|>e^{\delta/2}} \frac{C}{|z|^k} d\mu(z) \\ &= \int_{|z|>e^{\delta/2}} \frac{C}{|z|^5} \frac{1}{1-\frac{1}{|z|}} d\mu(z) \\ &\leq \frac{C}{1-e^{\delta/2}} \int_{|z|>e^{\delta/2}} \frac{1}{|z|^5} d\mu(z) < \infty, \end{aligned}$$

the last integral being finite by Proposition 4.22. □

Finally, letting (p,q) go to infinity in the equation of Lemma 5.6, using the continuity of the complex length spectrum, the continuity of the volume, and the fact that the lengths of the core geodesics $\varphi^i_{p/q}$ go to zero, we deduce the following generalization of Theorem 5.4 for even dimensions n. In the following theorem we have also included the odd dimensional case, as it is handled in a similar way.

Theorem 5.8. *Let M be a complete hyperbolic 3-manifold of finite volume. Then for $m \geq 3$*

$$\log \left| \frac{\mathcal{T}_{2m+1}(M)}{\mathcal{T}_5(M)} \right| = \sum_{k=3}^{m} \log |R_{2k}(k)| - \frac{1}{\pi} \operatorname{Vol} M \left(m(m+1) - 6 \right).$$

If in addition M is enriched with an acyclic spin structure, then for $m \geq 3$

$$\log \left| \frac{\mathcal{T}_{2m}(M,\eta)}{\mathcal{T}_4(M,\eta)} \right| = \sum_{k=2}^{m-1} \log \left| R_{2k+1} \left(k + \frac{1}{2} \right) \right| - \frac{1}{\pi} \operatorname{Vol} M (m^2 - 4).$$

The proof of Theorem 5.1 follows easily.

Proof of Theorem 5.1. Theorem 5.8 and Lemma 5.7 imply that

$$\lim_{n\to\infty} \frac{\log |\mathcal{T}_n(M,\eta)|}{n^2} = -\frac{\operatorname{Vol} M}{4\pi}.$$

□

6 Reidemeister torsion and length spectrum

Let (M, η) be a complete spin acyclic hyperbolic 3-manifold of finite volume. Theorem 5.8 shows that, for $n > 5$, $|\mathcal{T}_n(M, \eta)|$ is completely determined by the spin complex length spectrum of the manifold, its volume, and, depending on the parity of n, by $|\mathcal{T}_4(M, \eta)|$ or $|\mathcal{T}_5(M)|$. In this section, we want to study at what extent the sequence $\{|\mathcal{T}_n(M, \eta)|\}_{n \geq 4}$ determines the spin complex length spectrum of the manifold. Notice that by Theorem 5.1 this sequence determines the volume of the manifold.

The first observation is that $\{|\mathcal{T}_n(M, \eta)|\}_{n \geq 4}$ cannot completely determine $\mu_{\rm sp}(M, \eta)$. Indeed consider $\overline{M}$, the manifold M with the orientation reversed. Thus, using the canonical one-to-one correspondence between spin structures on M and $\overline{M}$, it makes sense to consider both $\mathcal{T}_n(\overline{M}, \eta)$ and $\mu_{\rm sp}(\overline{M}, \eta)$. It is easily checked that $|\mathcal{T}_n(M, \eta)| = |\mathcal{T}_n(\overline{M}, \eta)|$, but in general $\mu_{\rm sp}(M, \eta) \neq \mu_{\rm sp}(\overline{M}, \eta)$ as we have

$$
\begin{aligned}
\mu_{\rm sp}(M, \eta) &= \sum_{\varphi \in \mathcal{PC}(M)} \delta_{e^{\lambda(\varphi)/2}}, \\
\mu_{\rm sp}(\overline{M}, \eta) &= \sum_{\varphi \in \mathcal{PC}(M)} \delta_{e^{\overline{\lambda(\varphi)}/2}},
\end{aligned}
$$

where $\lambda(\varphi)$ is the spin complex length function of (M, η). Of course, it may happen that $\mu_{\rm sp}(M, \eta) = \mu_{\rm sp}(\overline{M}, \eta)$, for instance if M has an orientation reversing isometry. The following theorem shows that $\{|\mathcal{T}_n(M, \eta)|\}_{n \geq 4}$ determines $\mu_{\rm sp}(M, \eta)$ up to complex conjugation.

Theorem 6.1. *Let $(M_1, \eta_1), (M_2, \eta_2)$ be two complete spin acyclic hyperbolic 3-manifolds of finite volume. If there exists N such that for all $n \geq N$, $|\mathcal{T}_n(M_1, \eta_1)| = |\mathcal{T}_n(M_2, \eta_2)|$, then we have,*

$$\mu_{\rm sp}(M_1, \eta_1) + \mu_{\rm sp}(\overline{M_1}, \eta_1) = \mu_{\rm sp}(M_2, \eta_2) + \mu_{\rm sp}(\overline{M_2}, \eta_2).$$

In particular, M_1 and M_2 have the same (real) prime length spectrum.

As a corollaries of Theorem 6.1 and Theorem 5.8 we deduce the following results.

Corollary 6.2. *If the hypotheses of the above theorem hold, then for all $n \geq 4$,*

$$|\mathcal{T}_n(M_1, \eta_1)| = |\mathcal{T}_n(M_2, \eta_2)|.$$

Proof. If $|\mathcal{T}_n(M_1, \eta_1)| = |\mathcal{T}_n(M_2, \eta_2)|$ for all $n \geq N$, then Theorem 6.1 implies that the right hand side of the two equations of Theorem 5.8 are equal for all $m \geq 3$. Taking $m \geq N/2$, this shows that for $n = 4, 5$, $|\mathcal{T}_n(M_1, \eta_1)| = |\mathcal{T}_n(M_2, \eta_2)|$, and hence the same is true for all $n \geq 4$. ∎

Corollary 6.3. *Assume (M, η) is a closed spin hyperbolic manifold. Then the knowledge of $|\mathcal{T}_n(M, \eta)|$ for $n \geq N$ is equivalent to the knowledge of $\mu_{\rm sp}(M, \eta)$ up to conjugation.*

Proof. The spin complex length spectrum $\mu_{\mathrm{sp}}(M,\eta)$ determines the Ruelle zeta function $R^M_{\rho_n}(s)$. If we know it only up to conjugation, then we know

$$F(s) := R^M_{\rho_n}(s) R^{\overline{M}}_{\rho_n}(s).$$

Notice that $R^{\overline{M}}_{\rho_n}(s) = \overline{R^M_{\rho_n}(s)}$, since they are meromorphic functions which agree for $\mathrm{Re}(s) > 2 + n/2$. In particular, we have

$$F(0) = R^M_{\rho_n}(0) R^{\overline{M}}_{\rho_n}(0) = |R^M_{\rho_n}(0)|^2 = |\tau_n(M,\eta)|^4,$$

where in the last equality we have used Wotzke's Theorem 5.3. The other implication follows from Theorem 6.1. □

As we will see, the same proof of Theorem 6.1 allows to conclude the following result.

Theorem 6.4. *Let M_1, M_2 be two complete hyperbolic 3-manifolds of finite volume. If there exists N such that for all $k \geq N$, $|\mathcal{T}_{2k+1}(M_1)| = |\mathcal{T}_{2k+1}(M_2)|$, then we have,*

$$\mu_{\mathrm{sp}}\, M_1 + \mu_{\mathrm{sp}}\, \overline{M_1} = \mu_{\mathrm{sp}}\, M_2 + \mu_{\mathrm{sp}}\, \overline{M_2}.$$

In particular, M_1 and M_2 have the same (real) prime length spectrum.

As noted in equation 5 of the preceding section, the values of the Ruelle functions that appear in Theorem 5.8 can be expressed in terms of $\mu_{\mathrm{sp}}(M,\eta)$. This shows that there is a close relationship between the quantities $\{\mathcal{T}_n(M,\eta)\}_{n\geq 4}$ and the result of integrating the function $\log(1 - z^{-n})$ respect to the measure

$$\nu = \mu_{\mathrm{sp}}(M,\eta) + \mu_{\mathrm{sp}}(\overline{M},\eta).$$

We are using the principal branch of the logarithm; notice that since the support of ν is contained in the exterior of the unit disk, that integral makes sense. Therefore, we are asking whether the measure ν is determined by the values of the integrals

$$\int_{|z|>1} \log(1 - z^{-n})\, d\nu(z), \quad n \geq 6.$$

This is of course a problem of analytical nature. It turns out that the answer to this question is positive. This is deduced from the following analytical result whose proof is given at the end of this section.

Proposition 6.5. *Let μ be a Radon complex-valued measure compactly supported in the interior of the unit disk D that satisfies the following conditions:*

1. $\mathbf{C} \setminus \operatorname{supp} \mu$ *is connected.*

2. $\operatorname{supp}\mu$ *has zero Lebesgue measure.*

3. There exists a positive integer N and a holomorphic function ψ on the open unit disk with $\psi(0) = 0$, $\psi'(0) = 1$ such that

$$\int_D \frac{\psi(z^n)}{z^N} d\mu(z) = 0.$$

for all $n \geq N$.

Then $\mu = 0$.

Proof of Theorem 6.1. We can assume that $N \geq 6$. Let us put $\mu_i = \mu_{\rm sp}(M_i, \eta_i)$ and $\overline{\mu}_i = \mu_{\rm sp}(\overline{M_i}, \eta_i)$, for $i = 1, 2$. From Theorem 5.8 we deduce that for $k \geq 3$,

$$\begin{aligned} \log\left|\frac{\mathcal{T}_{2k+3}(M_i)}{\mathcal{T}_{2k+1}(M_i)}\right| &= \log|R^{M_i}_{2k+2}(k+1)| - \frac{2(k+1)}{\pi}\operatorname{Vol} M_i \\ \log\left|\frac{\mathcal{T}_{2k+2}(M_i)}{\mathcal{T}_{2k}(M_i)}\right| &= \log\left|R^{M_i}_{2k+1}\left(k+\frac{1}{2}\right)\right| - \frac{2k+1}{\pi}\operatorname{Vol} M_i. \end{aligned}$$

By hypothesis, there exists N such that for all $n \geq N$, $|\mathcal{T}_n(M_1, \eta_1)| = |\mathcal{T}_n(M_2, \eta_2)|$. Then, by Theorem 5.1, we have $\operatorname{Vol} M_1 = \operatorname{Vol} M_2$. On the other hand,

$$\begin{aligned} \log|R^{M_i}_{2k+2}(k+1)| &= \int_{|z|>1} \log|1 - z^{-2(k+1)}| d\mu_i(z), \\ \log\left|R^{M_i}_{2k+1}\left(k+\frac{1}{2}\right)\right| &= \int_{|z|>1} \log|1 - z^{-2k-1}| d\mu_i(z). \end{aligned}$$

Therefore, for all $n \geq N+1$, we have

$$\int_{|z|>1} \log|1 - z^{-n}| d\mu_1(z) = \int_{|z|>1} \log|1 - z^{-n}| d\mu_2(z). \tag{7}$$

On the other hand,

$$\int_{|z|>1} 2\log|1 - z^{-n}| d\mu_i(z) = \int_{|z|>1} \log(1 - z^{-n}) d\mu_i(z) + \int_{|z|>1} \log(1 - z^{-n}) d\overline{\mu}_i(z).$$

For $i = 1, 2$, let ν_i be the image measure of $\mu_i + \overline{\mu}_i$ under the map $z \mapsto \frac{1}{z}$. Since $\operatorname{supp}\mu_i \subset \{z \mid |z| > 1\}$, equation 7 is equivalent to,

$$\int_{|z|<1} \log(1 - z^n) d\nu_1(z) = \int_{|z|<1} \log(1 - z^n) d\nu_2(z),$$

for all $n \geq N+1$. The measure ν_i is not Radon since any neighbourhood of the origin has infinite measure. However, by Proposition 4.22, $z^5\nu_i$ is finite. Hence, $\nu = z^{N+1}(\nu_1 - \nu_2)$ is a Radon measure that satisfies

$$\int_{|z|<1} \frac{\log(1-z^n)}{z^{N+1}} d\nu(z) = 0,$$

for all $n \geq N+1$. Hence, we can apply Proposition 6.5 with $\psi(z) = -\log(1-z)$ to conclude that $\nu = 0$, what implies $\mu_1 + \overline{\mu}_1 = \mu_2 + \overline{\mu}_2$, as we wanted to prove. $\square$

Proof of Theorem 6.4. By the same argument given in the proof of 6.1, the hypotheses of Theorem 6.4 imply that we know the values of the integrals

$$\int_{|z|>1} \log(1 - z^{-2k}) d(\mu_{\mathrm{sp}}(M,\eta) + \mu_{\mathrm{sp}}(\overline{M},\eta))(z),$$

where $\nu = \mu_{\mathrm{sp}}(M,\eta) + \mu_{\mathrm{sp}}(\overline{M},\eta)$, and η is some acyclic spin structure on M. Since $\mu_{\mathrm{sp}}\, M$ is the pushforward measure of $\mu_{\mathrm{sp}}(M,\eta)$ under the square map $z \mapsto z^2$, a change of variable yields,

$$\int_{|z|>1} \log(1 - z^{-2k}) d(\mu_{\mathrm{sp}}(M,\eta) + \mu_{\mathrm{sp}}(\overline{M},\eta))(z) = \int_{|z|>1} \log(1 - z^{-k}) d(\mu_{\mathrm{sp}}\, M + \mu_{\mathrm{sp}}\, \overline{M})(z).$$

The proof then goes on as in the proof of Theorem 6.1. $\square$

6.1 A result on complex analysis

Let $H(D)$ be the space of holomorphic functions on the open unit disk endowed with the topology of the uniform convergence on compact sets. In what follows, $\psi(z)$ will denote a holomorphic function in $H(D)$ such that $\psi(0) = 0$ and $\psi'(0) = 1$. Then the following result holds.

Proposition 6.6. *For all $N \geq 1$, the linear span of $\left\{\frac{\psi(z^k)}{z^N}\right\}_{k \geq N}$ is dense in $H(D)$.*

As we have not been able to find this result in the literature, we provide a proof of it in this subsection.

Since the linear span of the monomials $\{z^n\}_{n\geq 0}$ is dense in $H(D)$, Proposition 6.6 is equivalent to say that for all $n \geq 0$ there exists a sequence $\{a_j^n\}_{j\geq N}$ of complex numbers such that

$$z^n = \sum_{k \geq N} a_k^n \frac{\psi(z^k)}{z^N},$$

with the right hand side converging uniformly on every compact set of D. If we express $\psi(z)$ as a power series, the above equality yields a linear system with $\{a_j^n\}_{j\geq N}$ as unknowns. Since

$\psi(0) = 0$ and $\psi'(0) = 1$, this system is lower triangular with ones in the diagonal, and hence it has a unique solution. The difficult point is to prove the convergence of the corresponding sequence. In order to prove Proposition 6.6 we proceed in a different slightly way.

Let $H(D_R)$ be the space of holomorphic functions on the open disk $D_R = \{z \in \mathbf{C} \mid |z| < R\}$ endowed with the topology of the uniform convergence on compact sets. Consider the Bergman space on the disk of radius $R > 0$, that is,

$$A^2(D_R) = \left\{ f \in H(D_R) \mid \int_{D_R} |f(z)|^2 dA(z) < \infty \right\},$$

where $dA(z)$ is the usual area measure. It is well known that $A^2(D_R)$ is a Hilbert space respect to the inner product,

$$\langle f, g\rangle = \int_{D_R} f(z)\overline{g(z)}dA(z).$$

The reason to consider $A^2(D_R)$ is due to the following well known result (see [HKZ00]), and the fact that it is a Hilbert space.

Proposition 6.7. *If a sequence of functions $\{f_n\}$ in $A^2(D_R)$ converges to f in $A^2(D_R)$, then $\{f_n\}$ converges to f uniformly on each compact set of D_R.*

A basis of $A^2(D_R)$ is given by the following functions, which are just normalizations of the monomials $\{z^k\}$,

$$\phi_n(z) = \sqrt{\frac{n+1}{\pi}}\frac{z^n}{R^{n+1}}.$$

For $0 < R < 1$ and $\psi \in H(D)$ with $\psi(0) = 0$ and $\psi'(0) = 1$, consider the linear operator $A_\psi \colon A^2(D_R) \to A^2(D_R)$ defined by

$$\begin{aligned} A_\psi(1) &= 1, \\ A_\psi(z^n) &= \psi(z^n) = z^n + \sum_{j\geq 2} \psi_j z^{nj} \quad \text{for } n \geq 1. \end{aligned}$$

Proposition 6.8. *For $R > 1$, let $A_\psi = I + B_\psi$. Then $B_\psi \colon A^2(D_R) \to A^2(D_R)$ is Hilbert-Schmidt. In particular, B_ψ is compact and A_ψ is bounded.*

Proof. To be Hilbert-Schmidt means that $\sum_{n\geq 0}\langle B_\psi(\phi_n), B_\psi(\phi_n)\rangle < \infty$. Let us write B_ψ in terms of the basis $\{\phi_n\}$. We have, $B_\psi(\phi_0) = 0$, and for $n \geq 1$,

$$\begin{aligned} B_\psi(\phi_n) &= \sqrt{\frac{n+1}{\pi}}\frac{1}{R^{n+1}}\sum_{j\geq 2}\psi_j z^{nj} = \sqrt{\frac{n+1}{\pi}}\frac{1}{R^{n+1}}\sum_{j\geq 2}\psi_j\sqrt{\frac{\pi}{nj+1}}R^{nj+1}\phi_{nj} \\ &= \sum_{j\geq 2}\psi_j\sqrt{\frac{n+1}{nj+1}}R^{n(j-1)}\phi_{nj}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n\geq 0}\langle B_\psi(\phi_n), B_\psi(\phi_n)\rangle &= \sum_{n\geq 1}\sum_{j\geq 2}\psi_j^2\frac{n+1}{nj+1}R^{2n(j-1)} \leq \sum_{j\geq 2}\frac{2\psi_j^2}{j}\sum_{n\geq 1}R^{2n(j-1)} \\
&= \sum_{j\geq 2}\frac{2\psi_j^2}{j}\frac{R^{2(j-1)}}{1-R^{2(j-1)}} \leq \frac{4}{R^4(1-R^2)}\sum_{j\geq 2}\frac{\psi_j^2}{j+1}R^{2(j+1)}.
\end{aligned}$$

The last sum is exactly the π times the square of the norm in $A^2(D_R)$ of $\psi(z)-z$, which is finite. $\square$

Corollary 6.9. *The operator $A_\psi\colon A^2(D_R)\to A^2(D_R)$ is invertible.*

Proof. We have $A_\psi = I + B_\psi$, with B_ψ a compact operator. The matrix of the operator A_ψ in the basis $\{\phi_n\}$ is lower triangular, and has ones in the diagonal; hence, the kernel of A_ψ is trivial, and the Fredholm alternative implies that A_ψ is invertible. $\square$

Corollary 6.10. *The linear span of $\{1, \psi(z), \psi(z^2), \dots\}$ is dense in $H(D)$.*

Proof. Let us fix $g(z)\in H(D)$. Let $0<R<1$, and let $f_R(z)=\sum_{n\geq 0}a_n(R)z^n\in A^2(D_R)$ such that $A_\psi(f_R)=g$. Then the series $a_0(R)+\sum_{n\geq 1}a_n(R)\psi(z^n)$ converges to g in $A^2(D_R)$, so it converges uniformly to g in every compact contained in D_R. Since $f_R(z)$ also belongs to $A^2(D_{R'})$ for all $0<R'<R$, and is holomorphic, the coefficients $a_n(R)$ are independent of R, so $a_n(R)=a_n$. Hence $a_0+\sum_{n\geq 0}a_n\psi(z^n)$ converges to g in every compact set contained in the unit disk. This proves the result. $\square$

Proof of Proposition 6.6. Consider the following linear subspace of $H(D)$

$$C_N = \{\phi\in H(D) \mid \phi^{(j)}(0)=0, 0\leq j\leq N-1\}.$$

Since the derivative is continuous in $H(D)$, C_N is closed in $H(D)$. By the preceding corollary, it follows that C_N is the closure of $\{\psi(z^N), \psi(z^{N+1}), \dots\}$. On the other hand, C_N is homeomorphic to $H(D)$ via the linear map

$$\begin{aligned}
H(D) &\to C_N \\
\phi(z) &\mapsto z^N\phi(z).
\end{aligned}$$

Therefore, the closure of the linear span of $\{\frac{\psi(z^k)}{z^N}\}_{k\geq N}$ is the whole $H(D)$, as we wanted to prove. $\square$

6.2 An application of the Cauchy transform

Let μ be a Radon complex-valued measure compactly supported in the complex plane. The Cauchy transform of μ is defined by

$$\hat{\mu}(\zeta) = \int_{\mathbf{C}} \frac{d\mu(z)}{z - \zeta}.$$

We will need only the following properties of the Cauchy transform, see [Gam69]. We will denote by $\widehat{\mathbf{C}}$ the Riemann sphere.

Proposition 6.11. *The Cauchy transform has the following properties,*

1. *$\widehat{\mu}(\zeta)$ is analytic on $\widehat{\mathbf{C}} \setminus \operatorname{supp} \mu$ and vanishes at infinity.*

2. *If $\hat{\mu} = 0$ Lebesgue-almost everywhere, then $\mu = 0$.*

Proposition 6.12. *Let μ be a Radon complex-valued measure compactly supported in the complex plane that satisfies the following conditions:*

1. *$\mathbf{C} \setminus \operatorname{supp} \mu$ is connected.*

2. *$\operatorname{supp} \mu$ has zero Lebesgue measure.*

3. *For all $n \geq 0$, $\int_{\mathbf{C}} z^n d\mu(z) = 0$.*

Then $\mu = 0$.

Proof. Let $\hat{\mu}(\zeta)$ be the Cauchy transform of μ. We know that $\hat{\mu}(\zeta)$ is analytic on $\widehat{\mathbf{C}} \setminus \operatorname{supp} \mu$ and vanishes at ∞. Take $|\zeta|$ large enough so that $|z/\zeta| < 1$ for all $z \in \operatorname{supp} \mu$. Then, we have

$$\hat{\mu}(\zeta) = \int_{\mathbf{C}} \frac{d\mu(z)}{z - \zeta} = -\frac{1}{\zeta} \int_{\mathbf{C}} \frac{d\mu(z)}{1 - \frac{z}{\zeta}} = -\frac{1}{\zeta} \sum_{n \geq 0} \int_{\mathbf{C}} \frac{z^n}{\zeta^n} d\mu(\zeta) = 0$$

The last term being zero by hypothesis. Thus $\hat{\mu}$ is identically zero in a neighbourhood of ∞, and hence must be identically zero in the whole connected component of $\widehat{\mathbf{C}} \setminus \operatorname{supp} \mu$ containing ∞. Since $\mathbf{C} \setminus \operatorname{supp} \mu$ is connected, and $\operatorname{supp} \mu$ has zero Lebesgue measure, $\hat{\mu} = 0$ Lebesgue-almost everywhere. This implies that $\mu = 0$, as we wanted to prove. $\square$

Proof of Proposition 6.5. Apply Propositions 6.6 and 6.12. $\square$

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